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Abstract

Purpose – In this paper, the exact solutions of the Schlömilch's integral equation and its linear and non-linear generalized formulas with application are solved by using two efficient iterative methods. The Schlömilch's integral equations have many applications in atmospheric, terrestrial physics and ionospheric problems. They describe the density profile of electrons from the ionospheric for awry occurrence of the quasi-transverse approximations. The paper aims to discuss these issues.
Design/methodology/approach – First, the authors apply a regularization method combined with the standard homotopy analysis method to find the exact solutions for all forms of the Schlömilch's integral equation. Second, the authors implement the regularization method with the variational iteration method for the same purpose. The effectiveness of the regularization-Homotopy method and the regularization-variational method is shown by using them for several illustrative examples, which have been solved by other authors using the so-called regularization-Adomian method.
Findings – The implementation of the two methods demonstrates the usefulness in finding exact solutions.
Practical implications – The authors have applied the developed methodology to the solution of the Rayleigh equation, which is an important equation in fluid dynamics and has a variety of applications in different fields of science and engineering. These include the analysis of batch distillation in chemistry, scattering of electromagnetic waves in physics, isotopic data in contaminant hydrogeology and others.
Originality/value – In this paper, two reliable methods have been implemented to solve several examples, where those examples represent the main types of the Schlömilch's integral models. Each method has been accompanied with the use of the regularization method. This process constructs an efficient dealing to get the exact solutions of the linear and non-linear Schlömilch's integral equation which is easy to implement. In addition to that, the accompanied regularization method with each of the two used methods proved its efficiency in handling many problems especially ill-posed problems, such as the Fredholm integral equation of the first kind.

Keywords Regularization, Homotopy analysis method, Schlömilch's integral equation, Variational iteration method
Paper type Research paper

1. Introduction

Integral equations play an important role in mathematical analysis of natural and engineering problems. Due to this importance, researchers have been devising new and improved ways for finding accurate and efficient solutions to integral equations for decades.

In the field of ionosphere research, for example, ionization of electron and electrically charged atoms in the upper atmosphere is a process, which can be described by the Schlömilch's integral equations. Parand and Delkhosh (2017) have studied the linear and non-linear Schlomilch's integral equations and their generalized forms. They implemented the generalized fractional order of the Chebyshev orthogonal functions collocation method. Unz (1966) and Gething and Maliphant (1967) used the Schlömilch's integral equations to described the ionosphere electron profile in a closed form. De *et al.* (1994) derived an analytic

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form of the absorption index of the ionospheric plasma. They used it to build up the Schlomilch's integral equations. Gullberg and Budinger (1981) studied the single-photon emission computed tomography and had shown that if the point spread function is isotropic, then the Schlomilch's integral equations are obtained. Bougoffa *et al.* (2012) transformed the Schlomilch's integral equation in such a way that the Adomian decomposition method (ADM) could be applied to solve it.

Wazwaz (2015) studied the linear and the non-linear Schlomilch's integral equations and their generalized forms. He used the regularization method combined with the ADM to handle all forms of Schlomilch's integral equations. He found a way to develop a combined regularization-Adomian method that can be used to handle applied physics, applied mathematics and engineering problems.

In this paper, we derive and validate the homotopy analysis method (HAM) and variational iteration method (VIM) for the solution of the Schlomilch's integral equations. We compare the merits of the proposed methods against the ADM used by other authors.

2. Governing equation

In this paper, we develop two methods for the solution of the Schlömilch's integral equation. Let us write the Schlömilch's integral equation in the following way:

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} y(x \sin t)dt, \tag{1}$$

where $f(x)$ is a known function, which has a continuous differential coefficient, when $-\pi \leqslant x \leqslant \pi$. Unz (1966) derived the following form, which represents a unique solution:

$$y(x) = f(0) + x\int_0^{\pi/2} f'(x \sin t)dt, \tag{2}$$

where f' is a derivation for f with respect to the argument $\omega = x \sin t$. In the following, four types of the Schlömilch's integral equation models will be solved: the general model (1), the generalized Schlömilch's integral equation, the Schlömilch-type integral equation, where the function of sine will be replaced by the function of cosine in the general formula and the non-linear Schlömilch's integral equation, where a non-linear term $F(y)$ will be used instead of the linear term y (x sin t) or y (x cos t).

In order to find the solutions, we will first implement the HAM (Liao, 2003; Zadeh *et al.*, 2010) with the use of the regularization method (Delves and Walch, 1974; Cherruault and Seng, 1997; Tikhonov, 1963a, b; Phillips, 1962). Second, we will use the VIM (He, 2000, 2006; Wazwaz, 2011a) with the use of the regularization method for the same purpose.

The four considered forms of the Schlömilch's integral equation are as followed: the general model (1), the generalized Schlömilch's integral equation formula (Wazwaz, 2015; Parand and Delkhosh, 2017):

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} y\left(x \sin^r t\right)dt,\ r \geqslant 1, \tag{3}$$

the Schlömilch-type integral equation formula:

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} y(x \cos^r t)dt,\ r \geqslant 1 \tag{4}$$

and the non-linear Schlömilch's integral equation:

$$f(x) = \frac{2}{\pi} \int_0^{\pi/2} F(y(x \sin^r t))dt, r \geqslant 1. \tag{5}$$

Here, $F(y)$ in the non-linear case is given by terms such as $y(x \sin^r t)$ or $y^2(x \sin^r t)$ and $y^3(x \sin^r t)$, etc.

The Schlömilch integral equation is a Fredholm integral equation of the first kind. The general forms of the linear and non-linear Fredholm integral equation of the first kind are (Wazwaz, 2011a):

$$f(x) = \lambda \int_a^b K(x,t)y(t)dt, \ x \in \Lambda, \tag{6}$$

$$f(x) = \lambda \int_a^b K(x,t)F(y(t))dt, \ x \in \Lambda, \tag{7}$$

where Λ is a closed-bounded domain of x and $K(x, t)$ is a known function. The solution of linear and/or non-linear Fredholm integral equation requires finding a function $y(t)$, which appears only inside of the integration sign. This causes difficulties in solving such types of equations. The following examples of the Schlömilch's integral model have been previously solved by using the regularization method (Delves and Walch, 1974; Cherruault and Seng, 1997; Tikhonov, 1963a, b; Phillips, 1962) accompanied by the ADM (Adomian, 1994; Adomian and Rach, 1992; Wazwaz, 2009; Jerri, 1999). In this paper, the linear and non-linear Schlömilch's integral models are solved using the well-known HAM and VIM methods, each of them accompanied with the regularization method. The proposed methods are described and validated below.

3. The regularization method

The regularization method was introduced by Tikhonov (1963a, b) and Phillips (1962). The method transforms the Fredholm integral equation of the first kind into the second kind. Considering the linear Fredholm integral equation of the first kind:

$$f(x) = \int_a^b K(x,t)y(t)dt, \tag{8}$$

where a and b are constants, we can use the regularization method (Wazwaz, 2011a) to transform it into the following approximate formula of the Fredholm integral equation:

$$\alpha y_\alpha(x) = f(x) - \int_a^b K(x,t)y_\alpha(t)dt. \tag{9}$$

It is clear that Equation (9) is a Fredholm integral equation of the second kind, where α is a positive small regularization parameter. Equation (9) can be rewritten in the following way:

$$y_\alpha(x) = \frac{1}{\alpha} f(x) - \frac{1}{\alpha} \int_a^b K(x,t)y_\alpha(t)dt. \tag{10}$$

It is important to mention that Tikhonov (1963a, b), Wazwaz (2011b, 2015) and Phillips (1962) proved that when $\alpha \to 0$ the solution of Equation (10), which is $y_\alpha(x)$, will approximate the exact solution of Equation (8) $y(x)$, where:

$$\lim_{\alpha \to 0} y_\alpha(x) = y(x). \tag{11}$$

It is worth mentioning that the solution of the Fredholm integral equation of the first kind may not exist, since this type of problem can be an ill-posed problem. If such solution exists, then it may not be unique.

4. The two efficient iterative methods

4.1 The HAM

The basic idea of homotopy in topology has been applied in a qualified analytic method for solving many linear and non-linear problems. The HAM has been developed by Liao in 1992 (Liao, 2003; Zadeh *et al.*, 2010). The HAM was implemented successfully in many kinds of non-linear problems, both analytically and numerically, where it proved its high efficiency in finding exact and approximate solutions.

Let us review the method by considering the following non-linear differential equation:

$$N[y(x)] = 0, \tag{12}$$

where N represents a non-linear operator, x is the independent variable and $y(x)$ represents an unknown function. For simplicity, all initial or boundary conditions will be ignored. Liao sets the so-called 0th-order deformation equation:

$$(1-q)L\left[\varphi(x;q)-y_0(x)\right] = qhH(x)N[\varphi(x;q)], \tag{13}$$

where $q \in [0, 1]$, which represents an embedding parameter. h represents a non-zero auxiliary parameter, $H(x)$ is the non-zero auxiliary function, L is the auxiliary linear operator, $\varphi(x; q)$ represents an unknown function and $y_0(x)$ denotes the initial guess for the function $y(x)$. It is important to mention that there is a great freedom in choosing the auxiliary parameters and functions in HAM. It is obvious that when $q = 0$ and $q = 1$, then we get:

$$\varphi(x;0) = y_0(x), \varphi(x;1) = y(x). \tag{14}$$

So, when the value of q increases from 0 to 1, the function of solution $\varphi(x; q)$ varies from the initial function $y_0(x)$ to the solution $y(x)$. When expanding the function $\varphi(x; q)$ into the Taylor series with respect to q, we get:

$$\varphi(x;q) = y_0(x) + \sum_{m=1}^{+\infty} y_m(x)q^m, \tag{15}$$

where $y_m(x)$ represents the following derivative:

$$y_m(x) = \frac{1}{m!} \left.\frac{\partial^m \varphi(x;q)}{\partial q^m}\right|_{q=0}. \tag{16}$$

If all the auxiliary parameters and functions, which were mentioned, are duly chosen, the series (15) will be converge at $q = 1$ and we will get the following result:

$$y(x) = y_0(x) + \sum_{m=1}^{+\infty} y_m(x). \tag{17}$$

Defining the following vector:

$$\vec{y}_n = \left\{y_0(x), y_0(x), \ldots, y_n(x)\right\}, \tag{18}$$

when differentiating Equation (13) m times with respect to q and then substituting $q = 0$ and finally dividing by $m!$, we get the following mth-order deformation equation:

$$L\left[y_m(x)-\chi_m y_{m-1}(x)\right] = hH(x)R_m\left(\vec{y}_{m-1}\right), \tag{19}$$

where R_m is equal to:

$$R_m\left(\vec{y}_{m-1}\right) = \frac{1}{(m-1)!}\left.\frac{\partial^{m-1}N[\varphi(x;q)]}{\partial q^{m-1}}\right|_{q=0}, \tag{20}$$

and:

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{21}$$

Implementing L^{-1} on both sides of Equation (19), we obtain:

$$y_m(x) = \chi_m y_{m-1}(x) + hL^{-1}\left[H(x)R_m\left(\vec{y}_{m-1}\right)\right]. \tag{22}$$

An iterative procedure can be used to obtain y_m for $m \geqslant 1$. The sum of all orders up to k yields:

$$y(x) = \sum_{m=0}^{k} y_m(x), \tag{23}$$

which is an accurate approximate function for the general differential Equation (12). Exact solution will be recovered, when $k \to +\infty$. If Equation (12) possesses a unique solution, then the HAM will recover this unique solution. When Equation (12) does not have a unique solution, then the HAM will give one solution among all available solutions.

4.2 The VIM

The VIM has been developed by He (He, 2000, 2006; Wazwaz, 2011a, b) in 1999. It has been proven that it is a reliable and effective method for many analytical and numerical purposes. VIM provides convergent successive approximations for the exact solution in a rapid way. This is unlike the ADM and HAM, which provide components for the exact solution. Furthermore, the VIM provides the exact solution in a series formula, which converges to the explicit formula, if such an exact solution exists.

Let us consider the general form of the Fredholm integral equation of the second kind:

$$y(x) = f(x) + \int_a^b K(x,t)F(y(t))dt, x \in \Lambda, \tag{24}$$

where Λ is a closed-bounded domain of x, f and F are known functions, K is the kernel of this integral equation and $y(x)$ is the unknown solution for this integral equation. In linear and non-linear cases, the Schlömilch's integral equation has just one function of x and t inside the integral sine, which is $F(y(x \sin t))$. So, we used the Leibnitz rule (Jerri, 1999; Kress, 1999; Wazwaz, 1997) for deriving the integrals of all forms of the Schlömilch's integral equation, since it is not possible to separate x outside the integral sign. Thus, we can use the following form instead of Equation (24):

$$y(x) = f(x) + \int_a^b F(y(x \sin t))dt, \ x \in \Lambda, \tag{25}$$

when differencing both sides of Equation (25) with respect to x using Leibnitz rule (Jerri, 1999; Kress, 1999; Wazwaz, 1997), we will get the following:

$$y'(x) = f'(x) + \int_a^b \frac{\partial}{\partial x} F(y(x \sin t))dt. \tag{26}$$

The correction functional for the above integro-differential Equation (26) is:

$$y_{n+1}(x) = y_{n+1}(x) + \int_0^x \lambda(t)\left(y_n'(t) - f'(t) - \int_a^b \frac{\partial}{\partial t} F(y(t \sin r))dr\right)dt, \tag{27}$$

where the VIM requires two main steps. First, determining the Lagrange multiplier $\lambda(t)$ by using the restricted variation and integration by parts as in He (2000, 2006) and Wazwaz (2011a). In general, $\lambda(t) = -1$ for the first-order intego-differential equation. After determining λ, the following iterative formula may be used without any restricted variations:

$$y_{n+1}(x) = y_{n+1}(x) - \int_0^x \left(y_n'(t) - f'(t) - \int_a^b \frac{\partial}{\partial t} F(y(t \sin r))dr\right)dt. \tag{28}$$

This formula is used to calculate the successive approximations $y_{n+1}(x)$, $n \geqslant 0$ for the solution $y(x)$. Although, any selective function can be used as the 0th approximation $y_0(x)$, it is preferable to use the given initial value $y(0)$ to select the 0th approximation $y_0(x)$. Finally, the solution will be given in the following form:

$$y(x) = \lim_{n \to \infty} y_n(x). \tag{29}$$

5. Solving the Schlömilch's integral equations by the HAM and VIM

5.1 The linear Schlömilch's integral equation

The regularization method described above transforms the linear Schlömilch's integral equation:

$$f(x) = \frac{2}{\pi} \int_0^{\pi/2} y(x \sin t)dt, x \in \Lambda, \tag{30}$$

into the following formula of the linear Schlömilch's integral equation of the second kind:

$$\alpha y_\alpha(x) = f(x) - \frac{2}{\pi} \int_0^{\pi/2} y_\alpha(x \sin t)dt. \tag{31}$$

Dividing by the positive small parameter α we obtain:

$$y_\alpha(x) = \frac{1}{\alpha} f(x) - \frac{1}{\alpha}\left(\frac{2}{\pi} \int_0^{\pi/2} y_\alpha(x \sin t)dt\right). \tag{32}$$

It is clear that when $\alpha \to 0$ approaches 0 the function y_α will converge to the solution $y(x)$ of Equation (30). Now, we can implement the mentioned iterative methods HAM and VIM to solve Equation (32). The following examples illustrate the usage of HAM and VIM to solve the Schlömilch's integral equation problems.

Example 1. Consider the following linear Schlömilch's integral equation:

$$1+\pi x^2=\frac{2}{\pi}\int_0^{\pi/2} y(x\sin t)dt,\ -\pi \leqslant x \leqslant \pi. \tag{33}$$

After applying the regularization method, Equation (33) is defined as follows:

$$y_\alpha(x)=\frac{1}{\alpha}\left(1+\pi x^2\right)-\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} y_\alpha(x\sin t)dt\right). \tag{34}$$

In the following, we will solve this equation will by HAM and VIM.

To implement the HAM for Equation (34), the same steps as stated in Section 3 will be applied. Therefore, the 0th approximation will be:

$$y_{\alpha 0}(x)=\frac{1}{\alpha}\left(1+\pi x^2\right). \tag{35}$$

Next we introduce the linear operator:

$$L[\varphi(x;q)]=\varphi(x;q). \tag{36}$$

Now, defining the non-linear operator in the form:

$$N[\varphi(x;q)]=\varphi(x;q)-\frac{1}{\alpha}\left(1+\pi x^2\right)+\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} \varphi(x\sin t;q)dt\right), \tag{37}$$

yields the following mth-order deformation equation:

$$L\left[y_{\alpha m}(x)-\chi_m y_{\alpha m-1}(x)\right]=hR_m\left(\vec{y}_{\alpha m-1}\right), \tag{38}$$

where:

$$R_m\left(\vec{y}_{\alpha m-1}\right)=y_{\alpha m-1}(x)-\left(1-\chi_m\right)\left(\frac{1}{\alpha}\left(1+\pi x^2\right)\right)+\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} y_{\alpha m-1}(x\sin t;q)dt\right),\ m\geqslant 0. \tag{39}$$

Using a symbolic manipulator we obtain the following series:

$$y_{\alpha 0}(x)=\frac{1}{\alpha}\left(1+\pi x^2\right),$$

$$y_{\alpha 1}(x)=\frac{h\left(2+\pi x^2\right)}{2\alpha^2},$$

$$y_{\alpha 2}(x)=h\left(\frac{h\left(4+\pi x^2\right)}{4\alpha^3}+\frac{h\left(2+\pi x^2\right)}{2\alpha^2}\right)+\frac{h\left(2+\pi x^2\right)}{2\alpha^2},$$

$$y_{\alpha 3}(x) = h\left(\frac{h(4+\pi x^2)}{4\alpha^3}+\frac{h(2+\pi x^2)}{2\alpha^2}\right)+\frac{h(2+\pi x^2)}{2\alpha^2}+h\left(h\left(\frac{h(4+\pi x^2)}{4\alpha^3}+\frac{h(2+\pi x^2)}{2\alpha^2}\right)\right.$$
$$\left.+\frac{h(2+\pi x^2)}{2\alpha^2}+\frac{h(h(8+\pi x^2)+2(1+h)(4+\pi x^2)\alpha)}{8\alpha^4}\right).$$
$$\vdots$$

Choosing $h = -1$ and introducing it into the series above, we obtain the following:

$$y_{\alpha 1}(x) = -\frac{1}{\alpha^2}-\frac{\pi x^2}{2\alpha^2},$$

$$y_{\alpha 2}(x) = \frac{1}{\alpha^3}+\frac{\pi x^2}{4\alpha^3},$$

$$y_{\alpha 3}(x) = -\frac{1}{\alpha^4}-\frac{\pi x^2}{8\alpha^4}.$$
$$\vdots$$

According to (23) we are able to derive:

$$y_\alpha(x) = \frac{1}{\alpha}\left(1-\frac{1}{\alpha}+\frac{1}{\alpha^2}-\frac{1}{\alpha^3}+\cdots\right)+\frac{\pi x^2}{\alpha}\left(1-\frac{1}{2\alpha}+\frac{1}{4\alpha^2}-\frac{1}{8\alpha^3}+\cdots\right). \tag{40}$$

In this expression, there are two geometric series, which we can sum up. The first one is $(1-(1/\alpha)+(1/\alpha^2)-(1/\alpha^3)+\cdots) = (1/(1-(-(1/\alpha))))$ and the second geometric series is $(1-(1/2\alpha)+(1/4\alpha^2)-(1/8\alpha^3)+\cdots) = (1/(1-(-(1/2\alpha))))$. Summing up, we obtain the following result:

$$y_\alpha(x) = \frac{1}{\alpha}\left(\frac{1}{1-\left(-\frac{1}{\alpha}\right)}\right)+\frac{\pi x^2}{\alpha}\left(\frac{1}{1-\left(-\frac{1}{2\alpha}\right)}\right) = \frac{1}{1+\alpha}+\frac{2\pi x^2}{1+2\alpha}. \tag{41}$$

When applying the limit, we are able to derive the exact solution:

$$y(x) = \lim_{\alpha\to 0} y_\alpha(x) = 1+2\pi x^2. \tag{42}$$

This solution is exactly the same as obtained by the ADM (Wazwaz, 2015).

In order to apply the VIM on Equation (34), both sides of (34) must be differentiated with respect to x. After differentiation we obtain the following:

$$y'_\alpha(x)-\frac{2}{\alpha}\pi x-\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2}\frac{\partial}{\partial x}y_\alpha(x\sin t)dt\right) = 0, \tag{43}$$

where the Leibnitz rule (Jerri, 1999; Kress, 1999; Wazwaz, 1997) will be used for differentiation of the integral of Equation (43). The correction functional for

Equation (43) is:

$$y_{\alpha n+1}(x) = y_{\alpha n}(x) - \int_0^x \left(y'_{\alpha n}(t) - \frac{2}{\alpha}\pi t + \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} \frac{\partial}{\partial t} y_{\alpha n}(t \sin r) dr \right) \right) dt, \tag{44}$$

where $\lambda = -1$. We select the initial approximation to be $y_{\alpha 0}(x) = 1 + (\pi x^2)/(\alpha)$. Then, we obtain the following series in a successive way:

$$\begin{aligned}
y_{\alpha 0}(x) &= 1 + \pi x^2, \\
y_{\alpha 1}(x) &= 1 - \frac{\pi x^2}{2\alpha^2} + \frac{\pi x^2}{\alpha}, \\
y_{\alpha 2}(x) &= 1 + \frac{\pi x^2}{4\alpha^3} - \frac{\pi x^2}{2\alpha^2} + \frac{\pi x^2}{\alpha}, \\
y_{\alpha 3}(x) &= 1 - \frac{\pi x^2}{8\alpha^4} + \frac{\pi x^2}{4\alpha^3} - \frac{\pi x^2}{2\alpha^2} + \frac{\pi x^2}{\alpha}, \\
y_{\alpha 4}(x) &= 1 + \frac{\pi x^2}{16\alpha^5} - \frac{\pi x^2}{8\alpha^4} + \frac{\pi x^2}{4\alpha^3} - \frac{\pi x^2}{2\alpha^2} + \frac{\pi x^2}{\alpha}. \\
&\vdots
\end{aligned}$$

Thus, according to Equation (29) we have following:

$$\begin{aligned}
y_\alpha(x) &= 1 + \frac{\pi x^2}{\alpha} - \frac{\pi x^2}{2\alpha^2} + \frac{\pi x^2}{4\alpha^3} - \frac{\pi x^2}{8\alpha^4} + \frac{\pi x^2}{16\alpha^5} - \cdots = 1 + \left(\frac{\frac{\pi x^2}{\alpha}}{1 - \left(-\frac{1}{2\alpha}\right)} \right) \\
&= \frac{1 + 2\pi x^2 + 2\alpha}{1 + 2\alpha},
\end{aligned}$$

where we used a summation rule for the geometric series. Taking a limit we are able to derive the exact solution:

$$y(x) = \lim_{\alpha \to 0} y_\alpha(x) = 1 + 2\pi x^2. \tag{45}$$

This result is the same as the exact solution obtained by the ADM (Wazwaz, 2015) and HAM above.

5.2 The generalized Schlömilch's integral equation

In order to deal with the generalized formulation of the Schlömilch's integral equation, the regularization method transforms the following equation of the first kind:

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} y(x \sin^n t) dt, \ x \in \Lambda, \tag{46}$$

into a new representation of the Schlömilch's integral equation of the second kind:

$$\alpha y_\alpha(x) = f(x) - \frac{2}{\pi}\int_0^{\pi/2} y_\alpha(x \sin^n t) dt. \tag{47}$$

Dividing by the positive small parameter α we obtain:

$$y_\alpha(x) = \frac{1}{\alpha} f(x) - \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} y_\alpha(x \sin^n t) dt \right). \tag{48}$$

It is clear that when $\alpha \to 0$ the function y_α will converge to the solution $y(x)$ of Equation (46). Both HAM and VIM will be implemented to solve Equation (48).

Example 2. Let us consider the following Schlömilch's integral equation:

$$x+3x^2 = \frac{2}{\pi}\int_0^{\pi/2} y\left(x\sin^2 t\right)dt, -\pi \leqslant x \leqslant \pi. \tag{49}$$

After the application of the regularization method, Equation (49) is:

$$y_\alpha(x) = \frac{1}{\alpha}\left(x+3x^2\right) - \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} y_\alpha\left(x\sin^2 t\right)dt\right). \tag{50}$$

This equation will be solved by HAM and VIM.

To implement the HAM for Equation (50), we follow the previously described algorithm. The 0th approximation is:

$$y_{\alpha 0}(x) = \frac{1}{\alpha}\left(x+3x^2\right) \tag{51}$$

and the linear operator is:

$$L[\varphi(x;q)] = \varphi(x;q). \tag{52}$$

Now, defining the non-linear operator in the following form:

$$N[\varphi(x;q)] = \varphi(x;q) - \frac{1}{\alpha}\left(x+3x^2\right) + \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} \varphi\left(x\sin^2 t;q\right)dt\right), \tag{53}$$

yields the following mth-order deformation equation:

$$L\left[y_{\alpha m}(x) - \chi_m y_{\alpha m-1}(x)\right] = hR_m\left(\vec{y}_{\alpha m-1}\right), \tag{54}$$

where:

$$R_m\left(\vec{y}_{\alpha m-1}\right) = y_{\alpha m-1}(x) - (1-\chi_m)\left(\frac{1}{\alpha}\left(x+3x^2\right)\right) + \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} y_{\alpha m-1}\left(x\sin^2 t;q\right)dt\right),\ m \geqslant 0. \tag{55}$$

Deriving, we get:

$$y_{\alpha 0}(x) = \frac{1}{\alpha}\left(x+3x^2\right),$$

$$y_{\alpha 1}(x) = \frac{hx(4+9x)}{8\alpha^2},$$

$$y_{\alpha 2}(x) = h\left(\frac{hx(16+27x)}{64\alpha^3} + \frac{hx(4+9x)}{8\alpha^2}\right) + \frac{hx(4+9x)}{8\alpha^2},$$

$$
\begin{aligned}
y_{\alpha 3}(x) &= h\left(\frac{hx(16+27x)}{64\alpha^3}+\frac{hx(4+9x)}{8\alpha^2}\right)+\frac{hx(4+9x)}{8\alpha^2}\\
&\quad +h\left(h\left(\frac{hx(16+27x)}{64\alpha^3}+\frac{hx(4+9x)}{8\alpha^2}\right)+\frac{hx(4+9x)}{8\alpha^2}\right.\\
&\quad \left.+\frac{hx(h\pi(64+81x)+8(1+h)\pi(16+27x)\alpha)}{512\pi\alpha^4}\right).\\
&\quad \vdots
\end{aligned}
$$

Introducing $h=-1$ to obtain the following series:

$$
\begin{aligned}
y_{\alpha 1}(x) &= -\frac{x(4+9x)}{8\alpha^2},\\
y_{\alpha 2}(x) &= \frac{x(16+27x)}{64\alpha^3},\\
y_{\alpha 3}(x) &= -\frac{x(64+81x)}{512\alpha^4}.\\
&\vdots
\end{aligned}
$$

According to (23), we recognize two geometric series:

$$
y_\alpha(x)=\frac{x}{\alpha}\left(1-\frac{1}{2\alpha}+\frac{1}{4\alpha^2}-\frac{1}{8\alpha^3}+\cdots\right)+\frac{3x^2}{\alpha}\left(1-\frac{3}{8\alpha}+\frac{9}{64\alpha^2}-\frac{27}{512\alpha^3}+\cdots\right), \tag{56}
$$

which after summation, yield:

$$
y_\alpha(x)=\frac{2}{1+2\alpha}x+\frac{24}{3+8\alpha}x^2. \tag{57}
$$

Finally, the exact solution is:

$$
y(x)=\lim_{\alpha\to 0} y_\alpha(x)=2x+8x^2. \tag{58}
$$

It is the same as the exact solution obtained by the ADM (Wazwaz, 2015).

In order to implement the VIM to Equation (50), both sides of the equation have to be differentiated with respect to x to give:

$$
y'_\alpha(x)-\frac{1}{\alpha}-\frac{6}{\alpha}x+\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2}\frac{\partial}{\partial x}y_\alpha\left(x\sin^2(3t)\right)dt\right)=0. \tag{59}
$$

The correction functional for Equation (59) is equal to:

$$
y_{\alpha n+1}(x)=y_{\alpha n}(x)-\int_0^x\left(y'_{\alpha n}(t)-\frac{1}{\alpha}-\frac{6}{\alpha}t+\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2}\frac{\partial}{\partial t}y_{\alpha n}\left(t\sin^2(3r)\right)dr\right)\right)dt, \tag{60}
$$

where $\lambda = -1$. After selecting the initial approximation as $y_{\alpha 0}(x) = (x)/(\alpha)+(3x^2)/(\alpha)$, we obtain in successive way the following series:

$$
\begin{aligned}
y_{\alpha 0}(x) &= \frac{2}{\alpha}x, \\
y_{\alpha 1}(x) &= -\frac{x}{2\alpha^2}-\frac{9x^2}{8\alpha^2}+\frac{x}{\alpha}+\frac{3x^2}{\alpha}, \\
y_{\alpha 2}(x) &= \frac{x}{4\alpha^3}+\frac{27x^2}{64\alpha^3}-\frac{x}{2\alpha^2}-\frac{9x^2}{8\alpha^2}+\frac{x}{\alpha}+\frac{3x^2}{\alpha}, \\
y_{\alpha 3}(x) &= -\frac{x}{8\alpha^4}-\frac{81x^2}{512\alpha^4}+\frac{x}{4\alpha^3}+\frac{27x^2}{64\alpha^3}-\frac{x}{2\alpha^2}-\frac{9x^2}{8\alpha^2}+\frac{x}{\alpha}+\frac{3x^2}{\alpha}, \\
y_{\alpha 4}(x) &= \frac{x}{16\alpha^5}+\frac{243x^2}{4096\alpha^5}-\frac{x}{8\alpha^4}-\frac{81x^2}{512\alpha^4}+\frac{x}{4\alpha^3}+\frac{27x^2}{64\alpha^3}-\frac{x}{2\alpha^2}-\frac{9x^2}{8\alpha^2}+\frac{x}{\alpha}+\frac{3x^2}{\alpha}. \\
&\vdots
\end{aligned}
$$

Summing the terms according to Equation (29) we recognize two geometric series:

$$y_\alpha(x) = \frac{x}{\alpha}\left(1-\frac{1}{2\alpha}+\frac{1}{4\alpha^2}-\frac{1}{8\alpha^3}+\cdots\right)+\frac{3x^2}{\alpha}\left(1-\frac{3}{8\alpha}+\frac{9}{64\alpha^2}-\frac{27}{512\alpha^3}+\cdots\right), \tag{61}$$

which may be summed to obtain:

$$y_\alpha(x) = \frac{2}{1+2\alpha}x+\frac{24}{3+8\alpha}x^2. \tag{62}$$

Finally, the exact solution is:

$$y(x) = \lim_{\alpha\to 0} y_\alpha(x) = 2x+8x^2. \tag{63}$$

It is the same exact solution as obtained by the ADM (Wazwaz, 2015).

5.3 The linear Schlömilch-type integral equation

We consider the linear Schlömilch-type integral equation similarly to formulations considered in previous sections. We use the regularization method to transform the following equation of the first kind:

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} y(x\cos t)dt, x \in \Lambda, \tag{64}$$

into the following formula of the Schlömilch-type integral equation of the second kind:

$$\alpha y_\alpha(x) = f(x)-\frac{2}{\pi}\int_0^{\pi/2} y_\alpha(x\cos t)dt. \tag{65}$$

By dividing with the positive small parameter α we obtain:

$$y_\alpha(x) = \frac{1}{\alpha}f(x)-\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} y_\alpha(x\cos t)dt\right). \tag{66}$$

When $\alpha \to 0$ the function y_α will converge to the solution $y(x)$ of Equation (64). The following examples illustrate the usage of HAM and VIM to solve Equation (66).

Example 3. Let us consider the following Schlömilch-type integral equation:

$$1+2x = \frac{2}{\pi}\int_0^{\pi/2} y(x\cos t)dt, \ -\pi \leqslant x \leqslant \pi. \tag{67}$$

By applying the regularization method, we can rewrite Equation (67) to get:

$$y_\alpha(x) = \frac{1}{\alpha}(1+2x) - \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} y_\alpha(x\cos t)dt\right). \tag{68}$$

To implement the HAM to Equation (68), the 0th approximation is:

$$y_{\alpha 0}(x) = \frac{1}{\alpha}(1+2x), \tag{69}$$

and the linear operator is:

$$L[\varphi(x;q)] = \varphi(x;q). \tag{70}$$

By defining the non-linear operator in the following form:

$$N[\varphi(x;q)] = \varphi(x;q) - \frac{1}{\alpha}(1+2x) + \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} \varphi(x\cos t;q)dt\right), \tag{71}$$

where the mth-order deformation equation is:

$$L\left[y_{\alpha m}(x) - \chi_m y_{\alpha m-1}(x)\right] = hR_m\left(\vec{y}_{\alpha m-1}\right), \tag{72}$$

$$R_m\left(\vec{y}_{\alpha m-1}\right) = y_{\alpha m-1}(x) - (1-\chi_m)\left(\frac{1}{\alpha}(1+2x)\right) + \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} y_{\alpha m-1}(x\cos t;q)dt\right), m \geqslant 0. \tag{73}$$

we are able to derive the approximant series in the following way:

$$\begin{aligned}
y_{\alpha 0}(x) &= \frac{1}{\alpha}(1+2x), \\
y_{\alpha 1}(x) &= \frac{h(\pi+4x)}{\pi\alpha^2}, \\
y_{\alpha 2}(x) &= h\left(\frac{h(\pi^2+8x)}{\pi^2\alpha^3} + \frac{h(\pi+4x)}{\pi\alpha^2}\right) + \frac{h(\pi+4x)}{\pi\alpha^2}, \\
y_{\alpha 3}(x) &= h\left(\frac{h(\pi^2+8x)}{\pi^2\alpha^3} + \frac{h(\pi+4x)}{\pi\alpha^2}\right) + \frac{h(\pi+4x)}{\pi\alpha^2} + h\left(h\left(\frac{h(\pi^2+8x)}{\pi^2\alpha^3} + \frac{h(\pi+4x)}{\pi\alpha^2}\right)\right. \\
&\quad \left. + \frac{h(\pi+4x)}{\pi\alpha^2} + \frac{h\left(h(\pi^3+16x) + (1+h)\pi(\pi^2+8x)\alpha\right)}{\pi^3\alpha^4}\right), \\
&\ \ \vdots
\end{aligned}$$

Taking $h=-1$ we obtain the following series:

$$\begin{aligned} y_{\alpha 1}(x) &= -\frac{1}{\alpha^2}-\frac{4x}{\pi\alpha^2}, \\ y_{\alpha 2}(x) &= \frac{1}{\alpha^3}+\frac{8x}{\pi^2\alpha^3}, \\ y_{\alpha 3}(x) &= -\frac{1}{\alpha^4}-\frac{16x}{\pi^3\alpha^4}. \\ &\vdots \end{aligned}$$

According to (23), we have recognize two geometric series:

$$y_\alpha(x) = \frac{1}{\alpha}\left(1-\frac{1}{\alpha}+\frac{1}{\alpha^2}-\frac{1}{\alpha^3}+\cdots\right)+\frac{2}{\alpha}x\left(1-\frac{2}{\pi\alpha}+\frac{4}{\pi^2\alpha^2}-\frac{8}{\pi^3\alpha^3}+\cdots\right), \tag{74}$$

which can be summed to give:

$$y_\alpha(x) = \frac{1}{1+\alpha}+\frac{2\pi}{2+\pi\alpha}x. \tag{75}$$

After taking the limit, the exact solution is:

$$y(x) = \lim_{\alpha\to 0} y_\alpha(x) = 1+\pi x. \tag{76}$$

Our results are the same as the exact solution obtained by the ADM (Wazwaz, 2015).

In order to implement the VIM on Equation (68), we differentiate both sides of the equation with respect to x to get:

$$y'_\alpha(x)-\frac{2}{\alpha}+\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2}\frac{\partial}{\partial x}y_\alpha(x\cos(t))dt\right) = 0. \tag{77}$$

The correction functional for Equation (77) is:

$$y_{\alpha n+1}(x) = y_{\alpha n}(x)-\int_0^x\left(y'_{\alpha n}(t)-\frac{2}{\alpha}+\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2}\frac{\partial}{\partial t}y_{\alpha n}(t\cos(r))dr\right)\right)dt, \tag{78}$$

where $\lambda=-1$. We select the initial approximation to be $y_{\alpha 0}(x)=1+(2x)/(\alpha)$, and obtain in successive way the following terms:

$$\begin{aligned} y_{\alpha 0}(x) &= 1+\frac{2x}{\alpha}, \\ y_{\alpha 1}(x) &= 1-\frac{4x}{\pi\alpha^2}+\frac{2x}{\alpha}, \\ y_{\alpha 2}(x) &= 1-\frac{4x}{\pi\alpha^2}+\frac{2x}{\alpha}-x\left(-\frac{4}{\pi\alpha^2}+\frac{4(-2+\pi\alpha)}{\pi^2\alpha^3}\right), \\ y_{\alpha 3}(x) &= 1-\frac{4x}{\pi\alpha^2}+\frac{2x}{\alpha}-x\left(-\frac{4}{\pi\alpha^2}+\frac{4(-2+\pi\alpha)}{\pi^2\alpha^3}\right)-x\left(-\frac{4(-2+\pi\alpha)}{\pi^2\alpha^3}+\frac{2(8+2\pi\alpha(-2+\pi\alpha))}{\pi^3\alpha^4}\right), \end{aligned}$$

$$
\begin{aligned}
y_{\alpha 4}(x) = {} & 1-\frac{4x}{\pi\alpha^2}+\frac{2x}{\alpha}-x\left(-\frac{4}{\pi\alpha^2}+\frac{4(-2+\pi\alpha)}{\pi^2\alpha^3}\right) \\
& -x\left(\frac{4(-2+\pi\alpha)\left(4+\pi^2\alpha^2\right)}{\pi^4\alpha^5}-\frac{2(8+2\pi\alpha(-2+\pi\alpha))}{\pi^3\alpha^4}\right) \\
& -x\left(-\frac{4(-2+\pi\alpha)}{\pi^2\alpha^3}+\frac{2(8+2\pi\alpha(-2+\pi\alpha))}{\pi^3\alpha^4}\right). \\
& \vdots
\end{aligned}
$$

Summing up according to Equation (29) we obtain two geometric series:

$$
y_\alpha(x) = \frac{1}{\alpha}\left(1-\frac{1}{\alpha}+\frac{1}{\alpha^2}-\frac{1}{\alpha^3}+\cdots\right)+\frac{2}{\alpha}x\left(1-\frac{2}{\pi\alpha}+\frac{4}{\pi^2\alpha^2}-\frac{8}{\pi^3\alpha^3}+\cdots\right), \tag{79}
$$

which may be summed up to give:

$$
y_\alpha(x) = \frac{1}{1+\alpha}+\frac{2\pi}{2+\pi\alpha}x. \tag{80}
$$

The limiting process yields the following exact solution:

$$
y(x) = \lim_{\alpha\to 0} y_\alpha(x) = 1+\pi x. \tag{81}
$$

The result is the same as the exact solution obtained by the ADM (Wazwaz, 2015).

5.4 The non-linear Schlömilch's integral equation

The non-linear Schlömilch's integral equation is:

$$
f(x) = \frac{2}{\pi}\int_0^{\pi/2} F(y(x\sin t))dt, \tag{82}
$$

where the function $F(y(x \sin t))$ represents the non-linearity of $y(x \sin t)$. For example, $y^j(x \sin t)$ or $y^j(x \cos t)$, … where $j \geqslant 2$.

In order to solve this non-linear Schlömilch's integral equation, we use the same analysis as in the linear case. So, Equation (82) is converted into a linear equation by using the following transformation:

$$
F(y(x\sin t)) = z(x\sin t), \quad y(x\sin t) = F^{-1}(z(x\sin t)). \tag{83}
$$

Then, the following equation is obtained:

$$
f(x) = \frac{2}{\pi}\int_0^{\pi/2} z(x\sin t)dt, \tag{84}
$$

which the regularization method converts into:

$$
z_\alpha(x) = \frac{1}{\alpha}f(x)-\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} z_\alpha(x\sin t)dt\right), \tag{85}
$$

where α is a positive small parameter. It is clear when $\alpha \to 0$ the function z_α will converge to the solution $z(x)$ of Equation (85). Now, we can implement the iterative methods (HAM and VIM) to solve the Schlömilch's integral equation of the second kind. The following example illustrates the application of HAM and VIM to solve a non-linear problem.

Example 4. Consider the following non-linear Schlömilch's integral equation:

$$5x^6 = \frac{2}{\pi}\int_0^{\pi/2} y^2(x\sin t)dt, \ -\pi \leqslant x \leqslant \pi. \tag{86}$$

After applying the transformation $z = y^2$ to convert the above equation into a linear equation, we obtain:

$$5x^6 = \frac{2}{\pi}\int_0^{\pi/2} z_\alpha(x\sin t)dt, \ -\pi \leqslant x \leqslant \pi. \tag{87}$$

Finally, by implementing the regularization method, Equation (87) transforms into:

$$z_\alpha(x) = \frac{1}{\alpha}(5x^6) - \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} z_\alpha(x\sin t)dt\right). \tag{88}$$

In order to develop HAM for Equation (88), the steps stated in Section 3 will be used. The 0th approximation is:

$$z_{\alpha 0}(x) = \frac{1}{\alpha}(5x^6), \tag{89}$$

and the linear operator is:

$$L[\varphi(x;q)] = \varphi(x;q). \tag{90}$$

Now, defining the non-linear operator in the form:

$$N[\varphi(x;q)] = \varphi(x;q) - \frac{1}{\alpha}(5x^6) + \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} \varphi(x\sin t;q)dt\right), \tag{91}$$

that the mth-order deformation equation will be:

$$L\left[z_{\alpha m}(x) - \chi_m z_{\alpha m-1}(x)\right] = hR_m\left(\vec{z}_{\alpha m-1}\right), \tag{92}$$

where:

$$R_m\left(\vec{z}_{\alpha m-1}\right) = z_{\alpha m-1}(x) - (1-\chi_m)\left(\frac{1}{\alpha}(5x^6)\right) + \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2} z_{\alpha m-1}(x\sin t;q)dt\right), \ m \geqslant 0. \tag{93}$$

we get the following series of terms:

$$z_{\alpha 0}(x) = \frac{1}{\alpha}(5x^6),$$

$$z_{\alpha 1}(x) = \frac{25hx^6}{16\alpha^2},$$

$$z_{\alpha 2}(x) = h\left(\frac{125hx^6}{256\alpha^3}+\frac{25hx^6}{16\alpha^2}\right)+\frac{25hx^6}{16\alpha^2},$$

$$z_{\alpha 3}(x) = h\left(\frac{125hx^6}{256\alpha^3}+\frac{25hx^6}{16\alpha^2}\right)+\frac{25hx^6}{16\alpha^2}+h\left(h\left(\frac{125hx^6}{256\alpha^3}+\frac{25hx^6}{16\alpha^2}\right)+\frac{25hx^6}{16\alpha^2}+\frac{125hx^6(16\alpha+h(5+16\alpha))}{4096\alpha^4}\right).$$

$$\vdots$$

Choosing $h=-1$, we obtain the following terms:

$$z_{\alpha 1}(x) = -\frac{25x^6}{16\alpha^2},$$

$$z_{\alpha 2}(x) = \frac{125x^6}{256\alpha^3},$$

$$z_{\alpha 3}(x) = -\frac{625x^6}{4096\alpha^4}.$$

$$\vdots$$

After summation, according to (23) we have:

$$z_\alpha(x) = \frac{5}{\alpha}x^6\left(1-\frac{5}{16\alpha}+\frac{25}{256\alpha^2}-\frac{125}{4096\alpha^3}+\cdots\right), \tag{94}$$

and by summing the geometric series we obtain:

$$z_\alpha(x) = \frac{80}{5+16\alpha}x^6. \tag{95}$$

The exact solution is obtained by the limiting process as:

$$z(x) = \lim_{\alpha\to 0} z_\alpha(x) = 16x^6. \tag{96}$$

After recalling the transformation $z=y^2$ the following exact solution is obtained:

$$z(x) = x^4, \quad y(x) = \pm 4x^3. \tag{97}$$

This result is the same exact solution as obtained by the ADM (Wazwaz, 2015).

To apply VIM on Equation (88), both sides of this equation will be differentiated with respect to x to get:

$$z'_\alpha(x) - \frac{30}{\alpha}x^5 + \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2}\frac{\partial}{\partial x}z_\alpha(x\sin(t))dt\right) = 0. \tag{98}$$

The correction functional for Equation (98) is:

$$z_{\alpha n+1}(x) = z_{\alpha n}(x) - \int_0^x\left(z'_{\alpha n}(t) - \frac{30}{\alpha}t^5 + \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2}\frac{\partial}{\partial t}z_{\alpha n}(t\sin(r))dr\right)\right)dt, \tag{99}$$

where $\lambda = -1$. The selected initial approximation is $z_{\alpha 0}(x) = (1/\alpha)(5x^6)$ and thus we obtain in a successive way the following terms:

$$\begin{aligned}
z_{\alpha 0}(x) &= \frac{1}{\alpha}(5x^6),\\
z_{\alpha 1}(x) &= -\frac{25x^6}{16\alpha^2} + \frac{5x^6}{\alpha},\\
z_{\alpha 2}(x) &= \frac{125x^6}{256\alpha^3} - \frac{25x^6}{16\alpha^2} + \frac{5x^6}{\alpha},\\
z_{\alpha 3}(x) &= -\frac{625x^6}{4096\alpha^4} + \frac{125x^6}{256\alpha^3} - \frac{25x^6}{16\alpha^2} + \frac{5x^6}{\alpha},\\
z_{\alpha 4}(x) &= \frac{3125x^6}{65536\alpha^5} - \frac{625x^6}{4096\alpha^4} + \frac{125x^6}{256\alpha^3} - \frac{25x^6}{16\alpha^2} + \frac{5x^6}{\alpha}.\\
&\vdots
\end{aligned}$$

After summation according to Equation (29) we have:

$$z_\alpha(x) = \frac{5}{\alpha}x^6\left(1 - \frac{5}{16\alpha} + \frac{25}{256\alpha^2} - \frac{125}{4096\alpha^3} + \cdots\right), \tag{100}$$

and hence by summing the geometric series we obtain:

$$z_\alpha(x) = \frac{80}{5+16\alpha}x^6. \tag{101}$$

Next, the limit process leads to the exact solution:

$$z(x) = \lim_{\alpha\to 0} z_\alpha(x) = 16x^6. \tag{102}$$

After recalling the transformation $z = y^2$, then the following exact solution will be obtained:

$$z(x) = x^4, \quad y(x) = \pm 4x^3. \tag{103}$$

This result is the same exact solution as obtained by Wazwaz (2015).

6. Application: the use of Schlömilch's integral equation for finding the continuous solution of the Rayleigh equation

When studying the periodic solutions for the following second-order non-linear ODE q (Ponzo and Wax, 1972):

$$y''+F(y')+y=0, \tag{104}$$

a polar representation can be obtained from Equation (104) as in Ponzo and Wax (1972) in this form:

$$J(x)=\frac{2}{\pi}\int_0^{\pi/2}\cos\theta G(x\cos\theta)d\theta. \tag{105}$$

Considering $F(x)=x-(x^3)/(3)$ in Equation (104) or $G(x)=x-(x^3/3)$ in Equation (105) gives us the Rayleigh equation, which is given in the following form:

$$J(r)=\frac{1}{2\pi}\int_0^{2\pi}\left[r\cos\theta-\frac{r^3}{8}\cos^3\theta\right]d\theta. \tag{106}$$

The Rayleigh equation is an important equation in fluid dynamics and has a variety of applications in different fields of science and engineering. These include the analysis of batch distillation in chemistry (Liu and Peng, 2007), scattering of electromagnetic waves in physics (Madrazo and Maradudin, 1997), isotopic data in contaminant hydrogeology (Abe and Hunkeler, 2006) and others. We can use the methodology developed in this paper for the solving of the Schlömilch's integral Equation (1) to find the solution of (105).

We start with the regularization method which transforms (105) into the formula of the Schlömilch's integral equation of the second kind:

$$G_\alpha(x)=\frac{1}{\alpha}J(x)-\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2}\cos\theta G_\alpha(x\cos\theta)d\theta\right), \tag{107}$$

where we use $J(x)=(x)/(8)(4-x^2)$ for the Rayleigh equation (Ponzo and Wax, 1972). Next, we use the implementation of the iterative methods HAM and VIM to the Schlömilch's integral equation to find the exact solution of Equation (107). Using HAM, we get the following components:

$$\begin{aligned}
G_{\alpha0}(x)&=\frac{x}{2\alpha}-\frac{x^3}{8\alpha},\\
G_{\alpha1}(x)&=\frac{hx}{4\alpha^2}-\frac{3hx^3}{64\alpha^2},\\
G_{\alpha2}(x)&=\frac{h^2x}{8\alpha^3}-\frac{9h^2x^3}{512\alpha^3}+\frac{hx}{4\alpha^2}+\frac{h^2x}{4\alpha^2}-\frac{3hx^3}{64\alpha^2}-\frac{3h^2x^3}{64\alpha^2},\\
G_{\alpha3}(x)&=\frac{h^3x}{16\alpha^4}-\frac{27h^3x^3}{4096\alpha^4}+\frac{h^2x}{4\alpha^3}+\frac{h^3x}{4\alpha^3}-\frac{9h^2x^3}{256\alpha^3}-\frac{9h^3x^3}{256\alpha^3}\\
&\quad+\frac{hx}{4\alpha^2}+\frac{h^2x}{2\alpha^2}+\frac{h^3x}{4\alpha^2}-\frac{3hx^3}{64\alpha^2}-\frac{3h^2x^3}{32\alpha^2}-\frac{3h^3x^3}{64\alpha^2}.\\
&\ \ \vdots
\end{aligned}$$

When $h=-1$, we obtain the following values:

$$G_{\alpha 1}(x) = -\frac{x}{4\alpha^2}+\frac{3x^3}{64\alpha^2},$$
$$G_{\alpha 2}(x) = \frac{x}{8\alpha^3}-\frac{9x^3}{512\alpha^3},$$
$$G_{\alpha 3}(x) = -\frac{x}{16\alpha^4}+\frac{27x^3}{4096\alpha^4}.$$
$$\vdots$$

Summation according to (23) yields:

$$G_\alpha(x) = \left(\frac{x}{2\alpha}-\frac{x}{4\alpha^2}+\frac{x}{8\alpha^3}-\cdots\right)-\left(\frac{x^3}{8\alpha}-\frac{3x^3}{64\alpha^2}+\frac{9x^3}{512\alpha^3}-\cdots\right), \tag{108}$$

two geometric series, such that the first one is $((x/2\alpha)-(x/4\alpha^2)+(x/8\alpha^3)-\cdots)=(x/2\alpha)/(1-(-(1/2\alpha)))=(x/1+2\alpha)$ and the second geometric series is $((x^3/8\alpha)-(3x^3/64\alpha^2)+(9x^3/512\alpha^3)-\cdots)=(x^3/8\alpha)/((1-(-(3/8\alpha))))=(x^3)/(3+8\alpha)$. Finally, we obtain:

$$G_\alpha(x) = \frac{x}{1+2\alpha}-\frac{x^3}{3+8\alpha}. \tag{109}$$

The exact solution is derived after taking a limit:

$$G(x) = \lim_{\alpha\to 0} G_\alpha(x) = x-\frac{x^3}{3}, \tag{110}$$

which is the exact solution of the Rayleigh equation (Ponzo and Wax, 1972).

The same exact solution can be obtained when applying the regularization-variational method. Here both sides of (107) must be differentiated with respect to x, to get:

$$G'_\alpha(x)-\frac{1}{2\alpha}+\frac{3x^2}{8\alpha}-\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2}\cos t\frac{\partial}{\partial x}G_\alpha(x\cos t)dt\right) = 0. \tag{111}$$

The Leibnitz rule (Jerri, 1999; Kress, 1999; Wazwaz, 1997) is used for differentiation the integral in Equation (111).

The correction functional for Equation (111) is:

$$G_{\alpha n+1}(x) = G_{\alpha n}(x)-\int_0^x\left(G'_{\alpha n}(t)-\frac{1}{2\alpha}+\frac{3t^2}{8\alpha}-\frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2}\cos r\left(\frac{\partial}{\partial t}G_{\alpha n}(t\cos r)\right)dr\right)\right)dt, \tag{112}$$

where $\lambda=-1$. The selected initial approximation is $G_{\alpha 0}(x)=(1/\alpha)((x/8)(4-x^2))$. Others are obtained in a successive way:

$$G_{\alpha 1}(x) = \frac{x}{2\alpha}-\frac{x^3}{8\alpha}-\frac{x}{4\alpha^2}+\frac{3x^3}{64\alpha^2},$$

$$G_{\alpha 2}(x) = \frac{x}{2\alpha}-\frac{x^3}{8\alpha}-\frac{x}{4\alpha^2}+\frac{3x^3}{64\alpha^2}+\frac{x}{8\alpha^3}-\frac{9x^3}{512\alpha^3},$$

$$G_{\alpha 3}(x) = \frac{x}{2\alpha} - \frac{x^3}{8\alpha} - \frac{x}{4\alpha^2} + \frac{3x^3}{64\alpha^2} + \frac{x}{8\alpha^3} - \frac{9x^3}{512\alpha^3} - \frac{x}{16\alpha^4} + \frac{27x^3}{4096\alpha^4},$$

$$\begin{aligned} G_{\alpha 4}(x) = {} & \frac{x}{2\alpha} - \frac{x^3}{8\alpha} - \frac{x}{4\alpha^2} + \frac{3x^3}{64\alpha^2} + \frac{x}{8\alpha^3} - \frac{9x^3}{512\alpha^3} \\ & - \frac{x}{16\alpha^4} + \frac{27x^3}{4096\alpha^4} + \frac{x}{32\alpha^5} - \frac{81x^3}{32768\alpha^5}. \\ & \vdots \end{aligned}$$

Summation according to Equation (29) gives:

$$\begin{aligned} G_\alpha(x) &= \left(\frac{x}{2\alpha} - \frac{x}{4\alpha^2} + \frac{x}{8\alpha^3} - \frac{x}{16\alpha^4} + \cdots\right) - \left(\frac{x^3}{8\alpha} - \frac{3x^3}{64\alpha^2} + \frac{9x^3}{512\alpha^3} - \frac{27x^3}{4096\alpha^4} + \cdots\right), \\ &= \left(\frac{\frac{x}{2\alpha}}{1 - \left(-\frac{1}{2\alpha}\right)}\right) - \left(\frac{\frac{x^3}{8\alpha}}{\left(1 - \left(-\frac{3}{8\alpha}\right)\right)}\right), \\ &= \frac{x}{1+2\alpha} - \frac{x^3}{3+8\alpha}, \end{aligned}$$

where this represents a summation of two geometric series. The exact solution may be derived by taking a limit:

$$G(x) = \lim_{\alpha \to 0} G_\alpha(x) = x - \frac{x^3}{3}. \tag{113}$$

This is the same exact solution of the Rayleigh equation as obtained by the HAM above.

7. Discussion

In this paper, two reliable analytic methods have been developed for solving several main types of the Schlömilch's integral equations. Each method has been accompanied with the use of a regularization method. The proposed algorithms are efficient way for deriving exact solutions of the linear and non-linear Schlömilch's integral equation. The use of the regularization method with each of the two used methods make them efficient in handling ill-posed problems, such as the Fredholm integral equation of the first kind. The solution of the Rayleigh equation, which was presented as an example of the application of our methodology, shows the way to apply our method and demonstrated its usefulness.

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