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Daftardar-Jafari method for solving nonlinear thin film flow problem

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ABSTRACT

The aim of this paper is to develop the Daftardar-Jafari iterative method (DJM) for a mathematical model that represents the nonlinear thin film flow of a non-Newtonian third-grade fluid on a moving belt with the aim to obtain an approximate solution of high accuracy. When applying the DJM there is no need to resort to any additional techniques such as evaluating Adomian's polynomials as in the Adomian decomposition method (ADM) or such as using Lagrange multipliers in the variational iteration method (VIM). The accuracy of our results is numerically verified by evaluating the functions of the error remainder and the maximal error remainders. In addition, these results are analyzed by comparing the accuracy of the DJM solutions with those of the fourth order Runge-Kutta method (RKM), ADM and VIM at the same parameter values. All the evaluations have been successfully performed in an iterative way by using the symbolic manipulator *Mathematica*[®].

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1. Introduction

In recent decades, the use of numerical methods has become a standard way to solve and evaluate different types of complex nonlinear problems. In this paper, we have proposed and developed an alternative approach – using iterative methods to find a solution with a high degree of accuracy. The iterative procedure leads to a series, which can be summed up to find an analytical formula, or it can form a suitable approximation. The error of the approximation can be controlled by properly truncating the series.

The subject of this study is about non-Newtonian fluids. Unlike Newtonian fluids, where the shear stress is linearly proportional to strain rate, the non-Newtonian fluid exhibit behaviour that is more complex. Examples of non-Newtonian fluids are salt solutions and molten polymers. Non-Newtonian fluids have been studied extensively in the last decades (Rajagopal, 1983) and are currently still a focus of many researchers (Bhatti, Zeeshan, & Ellahi, 2016; Rashidi, Bagheri, Momoniat, & Freidoonimehr, 2017; Ravnik & Skerget, 2015; Sheikholeslami & Zeeshan, 2017; Zeeshan & Atlas, 2017; Zeeshan et al., 2016).

Several iterative methods have been previously proposed for finding solutions of initial or boundary value problems. The most common are: the Adomian decomposition method (ADM) (Adomian, 1994; Siddiqui, Hameed, Siddiqui, & Ghorji, 2010), the variational iteration method (VIM) (He, 1999b), the

homotopy analysis method (HAM) (Liao, 2004), the homotopy perturbation method (HPM) (He, 1999a, 2000) and the differential transform method (DTM) (Bildik, Konuralp, Bek, & Kucukarslan, 2006; Zhou, 1986), etc.

In this paper, we implemented the Daftardar-Jafari method (DJM) (Daftardar-Gejji & Jafari, 2006) to solve the thin film flow of a third grade fluid on a moving belt. Our aim was to find an approximate solution without using any restricted assumptions. The DJM has been introduced for the first time by Varsha Daftardar-Gejji and Hossein Jafari in 2006. This iterative method has been successfully used to solve many kinds of problems. For instance; the application of DJM for solving different kinds of partial differential equations (Bhalekar & Daftardar-Gejji, 2008, 2012; Daftardar-Gejji & Bhalekar, 2010), solving the Laplace equation (Yaseen et al., 2013), solving the Volterra integro-differential equations with some applications for the Lane-Emden equations of the first kind (AL-Jawary & AL-Qaissy, 2015), solving the Fokker-Planck equation (AL-Jawary, 2016), Duffing equations (Al-Jawary & Al-Razaq, 2016) and calculating the steady-state concentrations of carbon dioxide absorbed into phenyl glycidyl ether solutions (Al-Jawary & Raham, 2016), and others. The thin film flow problem has been solved previously by the ADM and VIM (Siddiqui, Farooq, Haroon, & Babcock, 2012a), the semi-analytical iterative method by

Temimi and Ansari (TAM) (AL-Jawary, 2017) and other known iterative methods (Gul, Islam, Shah, Khan, & Shafie, 2014; Mabood, 2014; Mabood & Pochai, 2015; Moosavi, Momeni, Tavangar, Mohammadyari, & Rahimi-Esbo, 2016; Nemati, Ghanbarpour, Hajibabayi, & Hemmatnezhad, 2009; Sajid & Hayat, 2008; Shah, Pandya, & Shah, 2016; Siddiqui, Farooq, Haroon, Rana, & Babcock, 2012b). The following sections review the application of the DJM to solve the current problem and the validity of this method in finding the appropriate approximate solution.

2. The nonlinear thin film flow problem

In this section, we consider the thin film flow of non-Newtonian fluid on a moving belt (Siddiqui et al., 2012a). The flow is steady, laminar and uniform. The film thickness is also uniform. The following problem is governed by (Siddiqui et al., 2012a):

$$\frac{d^2w}{dx^2} + \frac{6(\beta_2 + \beta_3)}{\mu} \left(\frac{dw}{dx}\right)^2 \frac{d^2w}{dx^2} - \frac{\delta f}{\mu} = 0, \quad (1)$$

$$w(0) = V_0, \quad \frac{dw}{dx} = 0 \text{ at } x = \gamma, \quad (2)$$

where, w represents the fluid velocity, β_2 and β_3 are the material constants of the third-grade fluid, μ represents the dynamic viscosity, δ is the density, f is the acceleration with respect to gravity, γ is the uniform thickness of the film and V_0 is the speed of the belt.

The following dimensionless variables can be introduced as follows:

$$\tilde{x} = \frac{x}{\gamma}, \quad \tilde{w} = \frac{w}{V_0}, \quad \beta = \frac{(\beta_2 + \beta_3)V_0^2}{\mu\gamma^2}, \quad m = \frac{\delta f\gamma^2}{\mu V_0}. \quad (3)$$

The dimensionless form of the nonlinear boundary value problem of (1) and (2) with \sim removed is

$$\frac{d^2w}{dx^2} + 6\beta \left(\frac{dw}{dx}\right)^2 \frac{d^2w}{dx^2} - m = 0, \quad (4)$$

$$w(0) = 1, \quad \frac{dw}{dx} = 0 \text{ at } x = 1. \quad (5)$$

Since Equation (4) has two boundary conditions and since it is a second order nonlinear ODE it is considered to be a well-posed problem. By integrating Equation (4) twice and by using the boundary conditions given in (5), one can arrive to

$$\frac{dw}{dx} + 2\beta \left(\frac{dw}{dx}\right)^3 - mx = C, \quad (6)$$

where, C is the integration constant. When using the second condition shown in Equation (5) to calculate the integration constant in (6), the integration constant will be $C = -m$. Thus, the nonlinear

system of (4) and (5) can be represented with the following problem:

$$\frac{dw}{dx} + 2\beta \left(\frac{dw}{dx}\right)^3 - m(x-1) = 0, \quad w(0) = 1. \quad (7)$$

In the next sections, the basic steps of the DJM will be reviewed and applied to find an approximate solution for the problem presented by Equation (7).

3. The Daftardar-Jafari method

In order to demonstrate the steps of using the DJM; we first begin with considering the following general functional equation (Daftardar-Gejji & Jafari, 2006).

$$w = f + L(w) + N(w), \quad (8)$$

where, L denotes the linear operator, N is the nonlinear operator, f represents a given function and w is the solution for equation (8), which can be written as

$$w = \sum_{i=0}^{\infty} w_i. \quad (9)$$

Now, the following can be defined

$$G_0 = N(w_0), \quad (10)$$

$$G_m = N\left(\sum_{i=0}^m w_i\right) - N\left(\sum_{i=0}^{m-1} w_i\right), \quad (11)$$

so that $N(w)$ can be decomposed as

$$\begin{aligned} N\left(\sum_{i=0}^{\infty} w_i\right) &= \underbrace{N(w_0)}_{G_0} + \underbrace{[N(w_0 + w_1) - N(w_0)]}_{G_1} \\ &+ \underbrace{[N(w_0 + w_1 + w_2) - N(w_0 + w_1)]}_{G_2} \\ &+ \underbrace{[N(w_0 + w_1 + w_2 + w_3) - N(w_0 + w_1 + w_2)]}_{G_3} + \dots \end{aligned} \quad (12)$$

Moreover, the relation is defined with recurrence so that

$$w_0 = f, \quad (13)$$

$$w_1 = L(w_0) + G_0, \quad (14)$$

$$w_{m+1} = L(w_m) + G_m, \quad m = 1, 2, \dots \quad (15)$$

Since L represents a linear operator $\sum_{i=0}^m L(w_i) = L(\sum_{i=0}^m w_i)$, we may write

$$\begin{aligned} \sum_{i=1}^{m+1} w_i &= \sum_{i=0}^m L(w_i) + N\left(\sum_{i=0}^m w_i\right) \\ &= L\left(\sum_{i=0}^m w_i\right) + N\left(\sum_{i=0}^m w_i\right), \quad m = 1, 2, \dots \end{aligned} \quad (16)$$

So that,

$$\sum_{i=0}^{\infty} w_i = f + L\left(\sum_{i=0}^{\infty} w_i\right) + N\left(\sum_{i=0}^{\infty} w_i\right). \quad (17)$$

From the equation above, it is clear that $\sum_{i=0}^{\infty} w_i$ is the solution for Equation (8), where the functions w_i , $i = 0, 1, 2, \dots$ are obtained by the algorithm (13)–(15). The k -term series solution, which is given by $w = \sum_{i=0}^{k-1} w_i$, represents an approximate solution for Equation (17).

3.1. The convergence of the DJM

In 1922, the fixed point theorem has been proposed by Stefan Banach "1892–1945" (Banach, 1922). This theorem is very important in the field of functional analysis. Let us review it here.

Definition 3.1: (Banach, 1922) Let (X, d) be a metric space and let $N : X \rightarrow X$ be a Lipschitz continuous mapping then N is called a contraction mapping, if there exists a constant $0 \leq k < 1$ such that $d(N(x), N(y)) \leq k.d(x, y)$, for all $x, y \in X$.

Banach fixed point theorem: (Banach, 1922) Let (X, d) be a complete metric space and $N : X \rightarrow X$ be a contraction mapping then N admits a unique fixed point x_f in X , i.e. $N(x_f) = x_f$. Also x_f can be found as follows:

Starting with an arbitrary element x_0 in X and then defining a sequence $\{x_n\}$ as $x_n = N(x_{n-1})$, then $x_n \rightarrow x_f$.

Theorem 3.1: (Biazar & Ghazvini, 2009) Let X and Y be Banach spaces and $N : X \rightarrow Y$ be a contraction nonlinear mapping such that for some constant $0 \leq k < 1$

$$\|N(u) - N(u_f)\| \leq k\|u - u_f\|, \quad \forall u, u_f \in X,$$

Where, according to the fixed point theorem of Banach, there is a fixed point w such that $N(w) = w$, hence the generated terms by the DJM will be regarded as

$w_n = N(w_{n-1})$, $\lim_{n \rightarrow \infty} w_n = w$, and suppose that $w_0 \in B_r(w)$

where $B_r(w) = \{w^* \in X; \|w^* - w\| < r\}$ then we have the following statements:

1. $\|w_n - w\| \leq k^n \|w_0 - w\|$,
2. $w_n \in B_r(w)$,
3. $\lim_{n \rightarrow \infty} w_n = w$.

Proof: See (Biazar & Ghazvini, 2009).

In order to analyze the convergence of the DJM for solving the problem (8), we consider two solutions: w_{DJM} and w_{RKM} . The first is the approximate

solution, which is obtained by the DJM and the second is a numerical solution, which is obtained by using the Runge Kutta method (RKM) (AL-Jawary, 2017).

Now let $w_{RKM} - w_{DJM} = e$ be the error of the evaluated solutions w_{RKM} and w_{DJM} of (8). Let e satisfy (8) such that

$$e = f + L(e) + N(e). \quad (18)$$

Then the recurrence relation in (13)–(15) will take the following form

$$e_0 = f, \quad (19)$$

$$e_1 = L(e_0) + N(e_0), \quad (20)$$

$$e_{m+1} = L(e_m) + N\left(\sum_{i=0}^m e_i\right) - N\left(\sum_{i=0}^{m-1} e_i\right), \quad m = 1, 2, \dots \quad (21)$$

According to the nonlinear contraction mapping Theorem 3.1; if $\|w_n - w\| \leq k^n \|w_0 - w\|$, $0 \leq k < 1$ then

$$e_0 = f,$$

$$\|e_1\| = \|N(e_0)\| \leq k\|e_0\|,$$

$$\|e_2\| = \|N(e_0 + e_1) - N(e_0)\|$$

$$= \|N(e_1)\| \leq k\|e_1\| \leq k(k\|e_0\|) = k^2\|e_0\|,$$

therefore $\|e_2\| \leq k^2\|e_0\|$.

$$\|e_3\| = \|N(e_0 + e_1 + e_2) - N(e_0 + e_1)\|$$

$$= \|N(e_2)\| \leq k\|e_2\| \leq k(k\|e_1\|) \leq k(k(k\|e_0\|))$$

$$= k^3\|e_0\|,$$

therefore $\|e_3\| \leq k^3\|e_0\|$

In general, we have $\|e_{n+1}\| \leq k^{n+1}\|e_0\|$.

So that as $n \rightarrow \infty$ the error $e_{n+1} \rightarrow 0$ and that proves the convergence of the DJM for the general functional Equation (8). Please refer to (Bhalekar & Daftardar-Gejji, 2011; Hemeda, 2013) for more details.

4. Solving the governing problem by the DJM

In order to use the DJM to find an approximate solution for the problem (7), we have rewritten below this equation in the following way

$$\frac{dw}{dx} = m(x-1) - 2\beta \left(\frac{dw}{dx}\right)^3, \quad w(0) = 1. \quad (22)$$

By integrating (22) and using the given initial condition, we get

$$w = 1 - mx + m \frac{x^2}{2} - 2\beta \int_0^x \left(\frac{dw}{dt}\right)^3 dt. \quad (23)$$

We have $N(w) = -2\beta \int_0^x \left(\frac{dw}{dt}\right)^3 dt$ and $f = 1 - mx + m \frac{x^2}{2}$. Now, by applying the basic steps of the

DJM, we obtain the following set of approximations

$$\begin{aligned} w_0 &= 1 - mx + m \frac{x^2}{2}, \\ w_1 &= -\frac{1}{2}m^3(-1 + (-1 + x)^4)\beta, \\ w_2 &= \frac{1}{5}m^5(-2 + x)x\beta^2 \left(-64m^4x^7\beta^2 + 8m^4x^8\beta^2 + 2m^2x^5\beta(45 - 248m^2\beta) \right. \\ &\quad \left. + m^2x^6\beta(-15 + 232m^2\beta) + 10(3 - 6m^2\beta + 4m^4\beta^2) \right. \\ &\quad \left. - 20x(3 - 9m^2\beta + 8m^4\beta^2) - 40x^3(1 - 9m^2\beta + 16m^4\beta^2) \right. \\ &\quad \left. + 10x^2(7 - 33m^2\beta + 40m^4\beta^2) + 2x^4(5 - 120m^2\beta + 344m^4\beta^2) \right), \\ &\quad \vdots \end{aligned}$$

The series solution $w_{DJM,n}(x) = \sum_{i=0}^k w_i$ for Equation (22) can be derived by making the sum of the above components w_i obtained by the DJM. For ease and brevity, we mention the following series:

$$\begin{aligned} w_{DJM,2}(x) &= \sum_{i=0}^2 w_i = 1 + m \left(-1 + \frac{x}{2} \right) x \\ &\quad - 2\beta \left(-m^3x + \frac{3m^3x^2}{2} - m^3x^3 + \frac{m^3x^4}{4} + 6m^5x\beta \right. \\ &\quad - 15m^5x^2\beta + 20m^5x^3\beta - 15m^5x^4\beta + 6m^5x^5\beta \\ &\quad - m^5x^6\beta - 12m^7x\beta^2 + 42m^7x^2\beta^2 - 84m^7x^3\beta^2 \\ &\quad + 105m^7x^4\beta^2 - 84m^7x^5\beta^2 + 42m^7x^6\beta^2 - 12m^7x^7\beta^2 \\ &\quad + \frac{3}{2}m^7x^8\beta^2 + 8m^9x\beta^3 - 36m^9x^2\beta^3 + 96m^9x^3\beta^3 \\ &\quad - 168m^9x^4\beta^3 + \frac{1008}{5}m^9x^5\beta^3 - 168m^9x^6\beta^3 \\ &\quad \left. + 96m^9x^7\beta^3 - 36m^9x^8\beta^3 + 8m^9x^9\beta^3 - \frac{4}{5}m^9x^{10}\beta^3 \right). \end{aligned} \quad (24)$$

In the next subsection, we present the difference between the approximate solution of the DJM and the three standard variational iteration algorithms (He, Wu, & Austin, 2010; He, 2012).

4.1. The VIM algorithms

As in (He et al., 2010; He, 2012), the following form of nonlinear equation has been considered

$$Lw + Nw = 0, \quad (25)$$

where, L and N are the linear and nonlinear operators of this equation, respectively.

According to the VIM (He, 1999b), three variational iterative algorithms can be applied for solving the current nonlinear problem (7) (He, 2012).

Variational iteration algorithm-I:

$$w_{n+1}(x) = w_n(x) + \int_{x_0}^x \lambda \{Lw_n(t) + Nw_n(t)\} dt. \quad (26)$$

Variational iteration algorithm-II:

$$w_{n+1}(x) = w_0(x) + \int_{x_0}^x \lambda Nw_n(t) dt. \quad (27)$$

Variational iteration algorithm-III:

$$w_{n+2}(x) = w_{n+1}(x) + \int_{x_0}^x \lambda \{Nw_{n+1}(t) - Nw_n(t)\} dt, \quad (28)$$

Where, the Lagrange multiplier λ has been systematically explained in (He, 1999b). In general, when applying the VIM for solving (7), one selects $\lambda_3 = -1$ and the nonlinear operator is $Nw(x) = 2\beta \left(\frac{dw}{dx} \right)^3$. The employed functional when applying the variational iterative algorithm-I for solving (7) finally takes in the following form:

$$\begin{aligned} w_{1,n+1}(x) &= w_{1,n}(x) \\ &\quad - \int_0^x \left(\frac{dw_{1,n}}{dt} + 2\beta \left(\frac{dw_{1,n}}{dt} \right)^3 - m(t-1) \right) dt, \end{aligned} \quad (29)$$

where, $w_{1,0}(x) = 1$ and the other iterations are:

$$\begin{aligned} w_{1,1}(x) &= 1 - mx + \frac{mx^2}{2}, \\ w_{1,2}(x) &= 1 - mx + \frac{mx^2}{2} - \frac{1}{2}m^3(-1 + (-1 + x)^4)\beta, \\ w_{1,3}(x) &= 1 - mx + \frac{mx^2}{2} - \frac{1}{2}m^3(-1 + (-1 + x)^4)\beta \\ &\quad - 12m^5x\beta^2 + 30m^5x^2\beta^2 - 40m^5x^3\beta^2 + 30m^5x^4\beta^2 \\ &\quad - 12m^5x^5\beta^2 + 2m^5x^6\beta^2 + 24m^7x\beta^3 - 84m^7x^2\beta^3 \\ &\quad + 168m^7x^3\beta^3 - 210m^7x^4\beta^3 + 168m^7x^5\beta^3 \\ &\quad - 84m^7x^6\beta^3 + 24m^7x^7\beta^3 - 3m^7x^8\beta^3 - 16m^9x\beta^4 \\ &\quad + 72m^9x^2\beta^4 - 192m^9x^3\beta^4 + 336m^9x^4\beta^4 \\ &\quad - \frac{2016}{5}m^9x^5\beta^4 + 336m^9x^6\beta^4 - 192m^9x^7\beta^4 \\ &\quad + 72m^9x^8\beta^4 - 16m^9x^9\beta^4 + \frac{8}{5}m^9x^{10}\beta^4, \\ &\quad \vdots \end{aligned}$$

and so on. When applying the variational iterative algorithm-II for solving (7) the form of the employed functional reads as:

$$w_{2,n+1}(x) = w_{2,0}(x) - \int_0^x \left(2\beta \left(\frac{dw_{2,n}}{dt} \right)^3 - m(t-1) \right) dt, \quad (30)$$

and the final form of the functional used in the application of algorithm-III to solve (7) is

$$\begin{aligned} w_{3,n+2}(x) &= w_{3,n+1}(x) \\ &\quad - \int_0^x \left(2\beta \left(\frac{dw_{3,n+1}}{dt} \right)^3 - 2\beta \left(\frac{dw_{3,n}}{dt} \right)^3 - m(t-1) \right) dt. \end{aligned} \quad (31)$$

The approximate terms obtained by Equations (29), (30) and (31) are all the same.

After simplifying both of the DJM series form, i.e. $w_{DJM,n}(x) = \sum_{i=0}^k w_i(x)$ and the n th iteration obtained by the VIM $w_{VIM,n}(x)$; we observe that:

$$w_{DJM,n}(x) = w_{VIM,n+1}(x). \quad (32)$$

Consider this example: when making a simplification for $w_{DJM,2}(x)$ mentioned in (24) and $w_{VIM,3}(x)$ we have:

$$\begin{aligned} w_{DJM,2}(x) &= w_{VIM,3}(x) = 1 + \frac{1}{2}m(-2+x)x \\ &\quad - \frac{1}{2}m^3x(-4+6x-4x^2+x^3)\beta \\ &\quad + 2m^5x(-6+15x-20x^2+15x^3-6x^4+x^5)\beta^2 \\ &\quad - 3m^7x(-8+28x-56x^2+70x^3-56x^4+28x^5 \\ &\quad - 8x^6+x^7)\beta^3 + \frac{8}{5}m^9x(-10+45x-120x^2 \\ &\quad + 210x^3-252x^4+210x^5-120x^6+45x^7 \\ &\quad - 10x^8+x^9)\beta^4. \end{aligned}$$

It is worth mentioning that the n th iteration $w_{VIM,n}(x)$ represents the approximate solution obtained by applying any of the three standard variational iteration algorithms (29), (30) and (31).

We used Mathematica, the symbolic computation and manipulation software in our calculations. To check the accuracy of this approximate solution, we have suggested the following error remainder function

$$\begin{aligned} ER_n(x) &= \frac{d}{dx} \left(\sum_{i=0}^n w_i \right) + 2\beta \left(\frac{d}{dx} \left(\sum_{i=0}^n w_i \right) \right)^3 \\ &\quad - m(x-1) = 0, \end{aligned} \quad (33)$$

with the maximal error remainder parameter

$$MER_n = \max_{0 \leq x \leq 1} |ER_n(x)|, \quad (34)$$

All the terms that involve β and its powers give the contribution for the non-Newtonian fluid. Moreover, when setting $\beta = 0$ in the approximations above, we can retrieve the exact solution for the current problem of the Newtonian viscous fluid.

5. Numerical simulations and results

When inserting the values of β and m in the approximate solution (24) we can get several approximate solutions. We have chosen $\beta = 0.5$ and $m = 0.3$ as suggested by (AL-Jawary, 2017; Siddiqui et al., 2012a). The approximations by the DJM for this case are

$$w_0 = 1 + 0.3 \left(-1 + \frac{x}{2} \right) x,$$

$$w_1 = -1.(-0.027x + 0.0405x^2 - 0.027x^3 + 0.00675x^4),$$

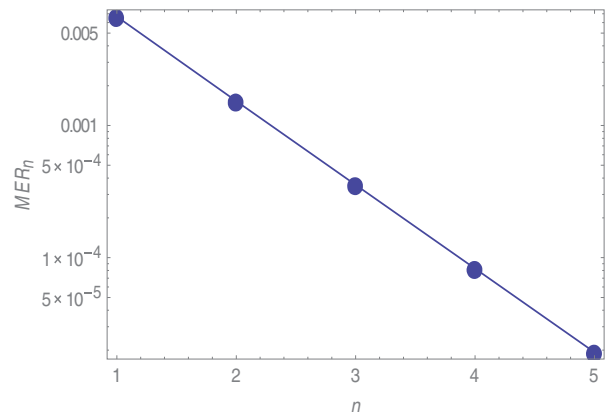


Figure 1. The logarithmic plots of MER_n by DJM when $\beta = 0.5$ and $m = 0.3$.

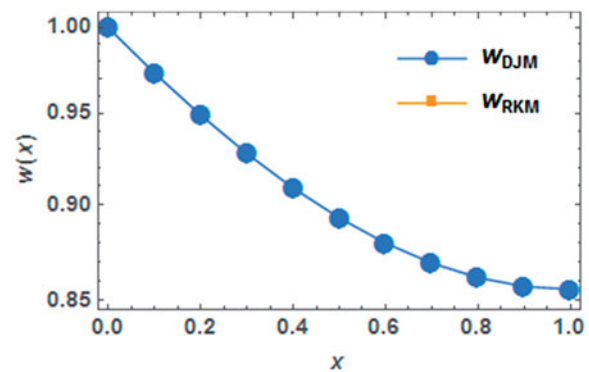


Figure 2. Comparison between the curves of the approximate series function by DJM and the numerical function which obtained by RKM for $0 \leq x \leq 1$ when $\beta = 0.5$ and $m = 0.3$.

$$\begin{aligned} w_2 &= 1.(-0.027x + 0.0405x^2 - 0.027x^3 + 0.00675x^4) \\ &\quad - 1.(-0.020346416999999995x + 0.024482776499999997x^2 \\ &\quad - 0.007056503999999997x^3 - 0.0061474680000000006x^4 \\ &\quad + 0.0031933115999999997x^5 + 0.0006680070000000001x^6 \\ &\quad - 0.0004199039999999999x^7 - 0.000006561000000000002x^8 \\ &\quad + 0.00001968299999999998x^9 - 0.0000019683x^{10}), \end{aligned}$$

⋮

The logarithmic plots of the maximum error remainder parameters MER_n , for $n = 1$ through 5 are shown in Figure 1 where an exponential rate of convergence can be seen. To show the validity of the DJM; Figure 2 shows the difference between the approximate solution, which is produced by the DJM and the numerical solution that is evaluated by using the RKM (AL-Jawary, 2017).

To show the validity for the DJM in reaching the best accuracy for the obtained approximate solutions, we have used the root mean square (RMS) norm to evaluate the difference between the solutions of the DJM and RKM. For this matter, the RMS

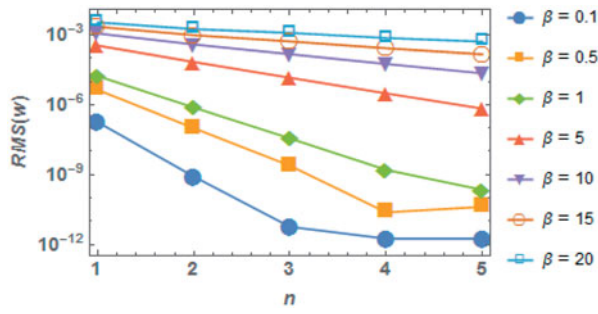


Figure 3. The curves of the RMS differences at different values of β when $m = 0.1$.

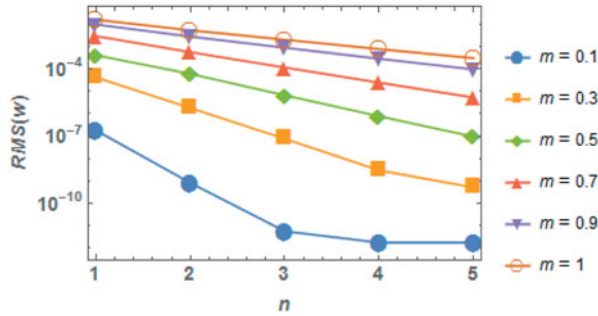


Figure 4. The curves of the RMS differences at different values of m when $\beta = 0.1$.

is given in the following form

$$RMS(w) = \sqrt{\frac{\sum (w_{DJM} - w_{RKM})^2}{\sum (w_{RKM})^2}}, \quad (35)$$

Figures 3 and 4 show the RMS differences versus n . We observe good convergence in the RMS curves as the value of n increases. At constant m note that the higher the β value, the larger the RMS difference, as shown in Figure 3. Also, keeping $\beta = 0.1$ with increasing the values of m will make the convergence poorer (Figure 4). In all cases we may conclude that the approximate DJM solution becomes more accurate whenever n increases. The rate of convergence with increasing n for the case of $\beta = 0.5$ and $m = 0.3$ was estimated using $\log(MER_4/MER_3)/\log(MER_3/MER_2) = 1.0$ proving linear convergence of the method.

6. Numerical comparisons

In this section, we present a comparison between our approximate solution obtained using the DJM and the approximate solutions obtained by previous studies using the ADM, VIM-I, VIM-II and VIM-III methods. In comparison, the ADM requires to evaluation the Adomian polynomials, which are computationally expensive. When comparing the DJM with the VIM-j; we find that there is no need for evaluating the Lagrange multipliers in DJM, which requires additional calculations when using the VIM-I, VIM-II

Table 1. The MER_5 for the solutions of the ADM, VIM-I and DJM for different values of β when $m = 0.3$.

β	ADM	VIM-I	DJM
0.1	1.39694×10^{-8}	4.06296×10^{-8}	2.11964×10^{-9}
0.2	8.59592×10^{-7}	1.15823×10^{-6}	1.17068×10^{-7}
0.3	9.43745×10^{-6}	7.88187×10^{-6}	1.16024×10^{-6}
0.4	0.0000512	0.0000299	5.71314×10^{-6}
0.5	0.000189	0.0000826	0.000019
1	0.010431	0.001694	0.000707

Table 2. The MER_5 for the solutions of the ADM, VIM-I and DJM for different values of m when $\beta = 0.5$.

m	ADM	VIM-I	DJM
0.1	1.39442×10^{-10}	7.56112×10^{-10}	2.22449×10^{-11}
0.2	1.06931×10^{-6}	1.27554×10^{-6}	1.42288×10^{-7}
0.3	0.000189099	0.0000826431	0.0000192222
0.4	0.00708853	0.00137525	0.000521409
0.5	0.108326	0.0107724	0.00584935

and VIM-III methods. Furthermore, the final solution in the DJM is based on the sum of resulting iterative terms. In contrast, the VIM-I approximate solution is obtained by taking the limit of the resulting successive approximations. Tables 1 and 2 present the error norm MER_5 for the solutions obtained by the ADM, VIM-I and DJM. It can be clearly seen that the best accuracy is obtained by the DJM numerical solution. The values of the MER_n for the fifth order approximate solutions is express smaller error of DJM in comparison to ADM and VIM-I.

Finally, when comparing the DJM with the other numerical methods, especially Runge–Kutta method (RKM); there is no need to use any type of truncation errors to measure the accuracy of the obtained approximate solution. There is no need for resorting to any discretization processes or determining the step size of the subintervals over the whole interval in the DJM. Furthermore, there is no need for making any round-off errors. The only limitation comes from the physical properties of the underlying problem. As values of the parameters β and m are increased the nature of the problem changes and thus the error obtained at a specific n increases. Changing of the parameters has an effect on convergence rate as well.

7. Conclusions

In this work, we have derived an approximate solution of the thin film flow of a non-Newtonian fluid by applying the Daftardar-Jafari iterative method. The DJM does not require any restricted assumptions, as they are required when using other iterative methods such as VIM or HAM. Furthermore, there is no need to resort to additional calculations such as evaluating Adomian polynomials as in the case of ADM. The differences and similarities between DJM

and VIM were explored in detail, highlighting the most important ones. By examining convergence properties of DJM for several parameter values of the thin film fluid flow problem, we observe good convergence properties. However, we did find that the choice of the parameters does have an effect on convergence.

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Disclosure statement

The authors declare that there is no conflict of interest.

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