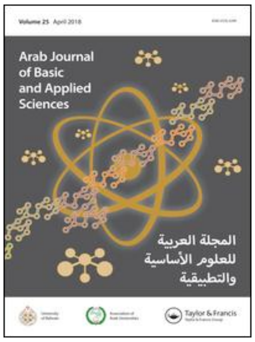

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Analytic and numerical solutions for linear and nonlinear multidimensional wave equations

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ABSTRACT

We develop three reliable iterative methods for solving the nonlinear 1D, 2D and 3D second-order wave equation and compare the results with a discretization-based solver. The iterative Tamimi–Ansari method (TAM), Daftardar–Jafari method (DJM) and the Banach contraction method (BCM) are used to obtain the exact solution for linear equations. For nonlinear equations and practical problems, however, one obtains the approximate solutions that converge to the exact solution, if one exists. The convergence analysis of the three methods is shown using the fixed-point theorem. The methods prove to be quite efficient and well suited to solve this kind of problems. We present several examples that demonstrate the accuracy and efficiency of the methods. We also compare the methods with a method based on discretization (Boundary Domain Integral Method (BDIM)). The BDIM uses a standard domain grid and discretizes the integral form of the governing equations. The iterative methods were developed with Mathematica® 10, while BDIM is a proprietary development.

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Approximate solution; boundary-domain integral method; exact solution; nonlinear wave equations 1D; 2D; 3D

1. Introduction

The wave equation is a partial differential equation for a scalar function that determines the wave propagation phenomena. It depends on time and one or more spatial variables. Apart from discretization approaches, such as finite element and finite volume approaches, many other methods have been proposed to solve the wave equation. For example, the Adomian decomposition method (Cheniguel, 2013), the optimal asymptotic homotopy method (Ullah et al., 2015), the Laplace Transform method (Oke, 2017), the high-order stereomodelling method (Tong, Yang, Hua, & Wang, 2013), the Variational Iteration Method (Biazar & Ghazvini, 2008), the new implicit second-order alternating direction method (Qin, 2009), the difference potential method (Britt, Tsynkov & Turkel, 2018), the fixed point iteration method, the Newton method (Shevchenko, 2008) and the local fractional variational iteration method (Jassim, 2015). In addition, there are many methods that provide an approximate solution for different types of differential equations (El-Ajou, Oqielat, Al-Zhour, Kumar & Momani, 2019; Ghanbari, Kumar, & Kumar, 2020; Goufo, Kumar & Mugisha, 2020; Jleli, Kumar, Kumar & Samet, in press; Kumar, Kumar, Abbas, Al Qurashi & Baleanu, 2020; Kumar, Kumar, Momani, Aldhaifallah, & Nisar, 2019; Kumar, Nisar, Kumar, Cattani & Samet, in press).

A new iterative method was proposed by Daftardar-Gejji and Jafari (DJM; Daftardar-Gejji & Jafari, 2006). The DJM has been used by many researchers for the treatment of linear and nonlinear ordinary differential equations (Bhalekar, & Daftardar-Gejji, 2012), partial differential equations of integer and fractional order (Daftardar-Gejji & Bhalekar, 2010; Yaseen, Samraiz, & Naheed, 2012), the Fokker–Planck equation (Al-Jawary, 2016), Korteweg–de-Vries equations (Ehsani, Hadi, Ehsani, & Mahdavi, 2013), the epidemic model and the prey and predator problems (Al-Jawary, 2014), Volterra integro-differential equations (Al-Jawary, & Al-Qaissy, 2015) etc.

A semi-analytical iterative technique was proposed by Temimi and Ansari (TAM) (Temimi & Ansari, 2011) to solve nonlinear problems. It has been used to solve many differential equations, such as second-order nonlinear ODEs that occur in physics (Al-Jawary, Adwan & Radhi, 2020), nonlinear Burgers advection-diffusion equations (Al-Jawary, Azeez & Radhi, 2018), Fornberg–Whitham equation (Almjeed, 2018), solving chemistry problems (Al-Jawary & Raham, 2017), Convective Straight and Radial Fins with temperature dependent thermal conductivity problems (Abdul Nabi & Al-Jawary, 2019), nonlinear

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thin film flow problems (Al-Jawary, 2017) and Fokker–Planck's equations (Al-Jawary, Radhi, & Ravnik, 2017). In addition, an alternative iterative method called Banach Contraction Principle (BCP) by Varsha Daftardar-Gejji and Sachin Bhalekar (Daftardar-Gejji, & Bhalekar, 2009) was proposed. This method considers fixed point theory as the main source of metrics. The BCP has been used to solve different types of differential and integral equations (Latif, 2014) such as nonlinear thin film flows of non-Newtonian fluids (Al-Jawary, Radhi & Ravnik, 2018).

The main objective of this article is to implement the three iterative methods TAM, DJM and BCM, to find an approximate solution of the wave equation. The iterative methods proposed in this article can be considered as alternatives to the established discretization approaches, such as finite differences, finite elements or the boundary-domain integral method. In this study, we compare the results of the iterative methods with the Boundary Domain Integral method (Ravnik & Tibaut, 2018) to assess their accuracy. There are also many analytical and numerical techniques that have proven to be effective and efficient in solving such problems (Bhatter, Mathur, Kumar & Singh, 2020; Goswami, Singh & Kumar, 2019; Gupta, Kumar & Singh, 2019; Kumar, Singh & Baleanu, 2018; Kumar, Singh, Purohit & Swroop, 2019).

This article was organized as follows: In Section 2 the standard formula of the wave equation is presented. In Section 3 the basic concepts of the proposed methods are shown. In Section 4 the convergence of the proposed methods is examined. In Section 5, the methods are demonstrated using several test cases. The conclusions are presented in the last section.

2. The formulation of the wave equations, approximate and numerical methods

Wave phenomena and the wave equation are extensively studied because of their importance for technical applications and for the understanding of many natural phenomena. Linear and nonlinear wave equations are studied by engineers, physicists and mathematicians (Biazar & Ghazvini, 2008; Keskin & Oturanc, 2010). In our study, we consider one-dimensional (1D), two-dimensional (2D) and three-dimensional (3D) nonlinear wave equations, which can be expressed for the 3D problem by the following formula

$$u_{tt} = \Delta u(x,y,z) + F(u) + f(x,y,z,t), \quad a < x,y,z < b, \quad t > 0, \tag{1}$$

with initial conditions

$$u(x,y,z,0) = f_1(x,y,z), \quad u_t(x,y,z,0) = f_2(x,y,z)$$

and appropriate Dirichlet type boundary conditions $F(u)$ can be linear or nonlinear.

In this section, we introduce the basic concepts of iterative methods TAM, DJM and BCM as well as the discretization type method Boundary Domain Integral (BDIM).

2.1. The basic idea of the TAM

Let us introduce the following nonlinear partial differential equation (Al-Jawary, Azeez et al., 2018)

$$L(u(x,t)) = N(u(x,t)) + g(x,t) = 0, \tag{2}$$

with the boundary conditions

$$B\left(u, \frac{du}{dt}\right) = 0,$$

where x is the independent variable, t is time, $u(x,t)$ is an unknown function, $g(x,t)$ the inhomogeneous term, L is a linear operator, N is a nonlinear operator and $B(.)$ is the boundary operator. We begin by assuming that $u_0(x,t)$ is an initial guess to solve the problem $u(x,t)$ and the solution algorithm starts by solving the following initial value problem

$$L(u_0(x,t)) = \; g(x,t), \;\text{ with } B\left(u_0, \frac{du_0}{dt}\right) = 0. \tag{3}$$

Next, an iterative procedure is set up to evaluate subsequent approximations $u_n(x,t)$ by solving the following problem

$$L(u_{n+1}(x,t)) = N(u_n(x,t)) + g(x,t), \; B\left(u_{n+1}, \frac{du_{n+1}}{dt}\right) = 0. \tag{4}$$

Then, the solution for Equation (4) is given by the following limit $u(x,t) = lim_{n\to\infty} u_n$.

2.2. The basic idea of the DJM

In this section, consider the following general functional equation (Yaseen et al., 2012)

$$u(x,t) = N(u(x,t)) + \; g(x,t), \tag{5}$$

where N is nonlinear operator and g is known function.

A solution $u(x,t)$ of Equation (5) is given by the following series

$$u(x,t) = \sum_{i=0}^{\infty} u_i. \tag{6}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}, \tag{7}$$

Considering Equations (6) and (7) we observe that Equation (5) is equivalent to

$$\sum_{i=0}^{\infty} u_i = g + N(u_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \tag{8}$$

We define the recurrence relation

$$u_0(x,t) = g(x,t), \tag{9}$$

$$u_1(x,t) = N(u_0(x,t)), \tag{10}$$

$$u_{m+1}(x,t) = N\left(\sum_{i=0}^{m} u_i(x,t)\right) - N\left(\sum_{i=0}^{m-1} u_i(x,t)\right), \; m = 1,2,3..., \tag{11}$$

and

$$\sum_{i=0}^{m-1} u_i(x,t) = N\left(\sum_{i=0}^{m} u_i(x,t)\right), \; m = 1,2,3.... \tag{12}$$

Finally, the solution is recovered by taking the following sum

$$u(x,t) = g(x,t) + \sum_{i=0}^{\infty} u_i(x,t). \tag{13}$$

2.3. The basic idea of the BCM

Consider the nonlinear functional equation (Al-Jawary, Radhi et al., 2018)

$$u(x,t) = N(u(x,t)) + g(x,t), \tag{14}$$

where $u(x,t)$ is an unknown function, N is nonlinear operator and $g(x,t)$ is a known function.

We define successive approximations as follows:

$$u_0(x,t) = g(x,t), \tag{15}$$

$$u_1(x,t) = u_0(x,t) + N(u_0(x,t)), \tag{16}$$

$$u_2(x,t) = u_0(x,t) + N(u_1(x,t)), \tag{17}$$

$$u_n(x,t) = u_0(x,t) + N(u_{n-1}(x,t)), \; n = 1,2,.... \tag{18}$$

2.4. The basic idea of the BDIM

The BDIM (Ravnik & Tibaut, 2018) is based on the fact that the fundamental solution of the problem is used to derive an integral formulation of the problem. The main advantage of BDIM is the use of the fundamental solution of the underlying physical problem as a weighting function in the derived integral formulation of the governing equations. Standard discretization methods such as FEM use shape functions to facilitate the derivation of the integral formulation and therefore do not take into account the physics of the phenomena. BDIM uses the fundamental solution and is able to detect physical effects on coarser meshes in comparison to FEM. The wave equation (Equation (1)) has a diffusive (Laplacian) operator and can be rewritten as follows

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = f, \tag{19}$$

where f is in general a nonlinear forcing term on the right hand side. The second order derivative over time is approximated using a second order finite difference approximation $u_{tt} = \frac{u_{-1}-2u-u_{+1}}{\Delta t}$ and included into the forcing term. We assume that initial conditions and mixed Dirichlet/Neumann boundary conditions are known. A time step Δt is introduced. Such a Poisson type equation can be written into integral form using a source point θ and the fundamental solution of the Laplace equation $u^{*}=1/(4\pi|r-\theta|)$ as

$$c(\theta)u(\theta) + \int_A u\nabla u^{*} dA = \int_A u^{*}\nabla u dA = \int_V u^{*} f dV \tag{20}$$

The free coefficient $c(\theta)$ is determined using the solid angle at the source point position. To write a discrete version of Equation (20), we have to interpolate the unknown function u and its flux ∇u over boundary and domain elements. In BDIM, the integral equation contains the boundary flux and the domain function. In our implementation of BDIM, we use quadratic interpolation of the function and linear interpolation of the boundary flux to achieve higher accuracy for simulation problems with high gradients in the solution. We use hexahedral domain elements and quadratic boundary elements. Finally the discrete version of Equation (20) can be written. A Gaussian quadrature algorithm is used to calculate the integrals.

A collocation scheme is used to write a system of linear equations for the unknown values of function and flux. The source point θ is placed in boundary and inner nodes. Since the boundary domain integral method requires domain discretization and since the matrix of domain integrals is full, we avoid excessive memory and computational time consumption by using a domain decomposition technique. Domain decomposition results in a sparse system of equations. In this work, we consider the subdomains as domain mesh elements. Connection between the subdomains is made by the fact that the function and the flux must be continuous across the boundaries of the subdomains. The described procedure leads to a sparse and overdetermined system of linear equations. We use a least squares solver with diagonal preconditioning to find the solution.

Since the problems considered in this article are nonlinear, we have set up an iteration procedure where we estimate the forcing using function values in the previous iteration. An under-relaxation of 0.1 was used to achieve the convergence. Since the problems considered are 1D, 2D and 3D and the BDIM method is written in 3D, we also used appropriate (zero flux) boundary conditions on the sidewalls. Further details on BDIM can be found in the work by Ravnik and Tibaut (2018) and references therein.

3. The convergence of the proposed iterative methods

In this section, we demonstrate the convergence of the proposed methods for the linear and nonlinear wave equation. We define new iterations as follows

$$\begin{aligned} v_0 &= u_0(x,t), \\ v_1 &= F[v_0], \\ v_2 &= F[v_0+v_1], \\ &\vdots \\ &. \\ v_{n+1} &= F[v_0+v_1+\ldots+v_n]. \end{aligned} \tag{21}$$

where F is the operator defined by

$$F[v_k] = S_k - \sum_{i=0}^{k-1} v_i(x,t), \;\; k=1,2,\ldots. \tag{22}$$

The term S_k represents the solution of the following problem

$$L(v_k(x,t)) + g(x,t) + N\Big(\sum_{i=0}^{k-1} v_i(x,t)\Big) = 0, \;\; k=1,\; 2,\ldots, \tag{23}$$

using the given conditions of the problem. In this way, we have $u(x,t) = lim_{n\to\infty} u_n(x,t) = \sum_{n=0}^{\infty} v_n$. So, the solution of the problem can be represented by using Equations (19) and (20) in the following series

$$u(x,t) = \sum_{i=0}^{\infty} v_i(x,t). \tag{24}$$

According to this procedure, sufficient conditions for convergence of our proposed iterative methods are presented below. The main results are stated in the following theorems.

Theorem 3.1. *Let F be an operator defined in Equation (22) from a Hilbert space H to H. The solution in a series formula $u_n(x,t) = \sum_{i=0}^{n} v_i(x,t)$ converges if $\exists\; 0 < r < 1$ such that $||F[v_0+v_1+\;\ldots+v_{i+1}]|| \le r||F[v_0+v_1+\ldots+v_i]||$ (that is $||v_{i+1}|| \le r||v_i||$) $\forall i = 0,1,2,\ldots.$*

This theorem is not only a special case of the Banach fixed-point theory, but it is a sufficient condition to study the convergence.

Proof. See (Odibat, 2010). □

Theorem 3.2. *Let the series solution $u(x,\;\; t) = \sum_{i=0}^{\infty} v_i(x,t)$ be convergent, then this series will represent the exact solution of the current nonlinear problem.*

Proof. See (Odibat, 2010). □

Theorem 3.3. *Suppose that the series solution $\sum_{i=0}^{\infty} v_i(x)$ presented by Equation (24) converges to the solution $u(x,t)$. If the truncated series $\sum_{i=0}^{n} v_i(x,t)$ is used as an approximation to the solution of the current problem, then the maximum error $E_n(x,t)$ is estimated by*

$$E_n(x,t) \le \frac{1}{1-r} r^{n+1} ||v_0||. \tag{25}$$

Proof. See (Odibat, 2010). □

Theorems 3.1 and 3.2 state that the solutions obtained by one of the presented methods, i.e. the relation (4) (for the TAM), the relation (11) (for the DJM), the relation (20) (for the BCM) or (21) converges to the exact solution under the condition $\exists\; 0 < r < 1$ such that $||F[v_0+v_1+\ldots+v_{i+1}]|| \le r\big|\big|F[v_0+v_1+\ldots+v_i]\big|\big|$ (that is $||v_{i+1}|| \le r||v_i||$) $\forall i = 0,1,2,\ldots$. In other words, for each i, if we define the parameters

$$\beta_i = \begin{cases} \dfrac{||v_{i+1}||}{||v_i||}, & ||v_i|| \ne 0 \\ 0, & ||v_i|| = 0 \end{cases} \tag{26}$$

then the series solution $\sum_{i=0}^{\infty} v_i(x,t)$ converges to the exact solution $u(x,t)$, when $0 \le \beta_i < 1$, $\forall i = 0,1,2,\ldots$. Furthermore, as shown in Theorem 3.3, the maximum truncation error is estimated to be $\big|\big|u(x,t) - \sum_{i=0}^{n} v_i\big|\big| \le \frac{1}{1-\beta}\beta^{n+1}||v_0||$, where $\beta = max\{\beta_i, i = 0,1,\ldots,n\}$.

4. Numerical examples

In this section, we proposed methods to solve several examples of the 1D, 2D, 3D linear and nonlinear wave equations.

Example 1. Consider the following 1D linear wave equation given by Wazwaz (2010)

$$u_{tt}(x,t) = u_{xx}(x,t) - 2, \tag{27}$$

with the following initial conditions:

$$u(x,0) = x^2, \;\; u_t(x,0) = \sin x.$$

Solution of Example 1 by TAM:

We first begin by solving the following initial problem as follows:

$$L(u(x,t)) = u_{tt}(x,t), \;\; N(u(x,t)) = u_{xx}(x,t), \;\; g(x,t) = -2 \tag{28}$$

The primary problem can be written as

$$L(u_0(x,t)) = -2, \;\; \text{with} \;\; u_0(x,0) = x^2, \;\; u_{0t}(x,0) = \sin x \tag{29}$$

We can get the following problems from the generalized general relationship

$$\begin{aligned} &L(u_{n+1}(x,t)) = g(x,t) + N(u_n(x,t)) = 0, \\ &u_{n+1}(x,0) = x^2, \;\; u_{(n+1)t}(x,0) = \sin x. \end{aligned} \tag{30}$$

We have

$$u_{0tt}(x,t) = -2, \tag{31}$$

by integrating both sides of Equation (31) twice from 0 to t, with $u_0(x,0) = x^2$, $u_{0t}(x,0) = \sin x$, we obtain

$$u_0 = -t^2 + x^2 + t\ \sin x,$$

In the same way, the rest of the iterations can be evaluated, the first iteration being

$$u_{1tt}(x,t) = u_{0xx}(x,t) - 2,\ \text{with}\ \ u_1(x,0) = x^2,\ u_{1t}(x,0) = \sin x. \tag{32}$$

Then, the solution for Equation (32) will be:

$$u_1 = t^2 + x^2 + t\ \sin x - \frac{1}{6}t^3\sin x,$$

We find the second iteration $u_2(x,t)$ by solving the following problem:

$$u_{2tt}(x,t) = u_{1xx}(x,t) - 2,\quad \text{with}\quad u_2(x,0) = x^2,\ u_{2t}(x,0) = \sin x. \tag{33}$$

Then, by solving Equation (33) we get

$$u_2 = x^2 + t\sin x - \frac{1}{6}t^3\sin x + \frac{1}{120}t^5\sin x.$$

Similarly, the third iteration $u_3(x,t)$ can be obtained by solving the following equation

$$u_{3tt}(x,t) = u_{2xx}(x,t) - 2,\ \text{with}\ u_3(x,0) = x^2,\ u_{3t}(x,0) = \sin x, \tag{34}$$

giving:

$$u_3 = x^2 + t\sin x - \frac{1}{6}t^3\sin x + \frac{1}{120}t^5\sin x - \frac{t^7\sin x}{5040}.$$

In a similar way, we get subsequent iterations as

$$u_4 = x^2 + t\ \sin x - \frac{1}{6}t^3\sin\ x + \frac{1}{120}t^5\sin x - \frac{t^7\sin x}{5040} + \frac{t^9\sin x}{362880},$$

$$u_5 = x^2 + t\ \sin x - \frac{1}{6}t^3\sin x + \frac{1}{120}t^5\sin x - \frac{t^7\sin x}{5040} + \frac{t^9\sin x}{362880} - \frac{t^{11}\sin x}{39916800}.$$

Finally, by taking the limit

$$u(x,t) = \lim_{n\to\infty} u_n$$

$$u(x,t) = x^2 + \sin x\left(t \pm \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{t^7}{5040} + \frac{t^9}{362880} - \frac{t^{11}\sin x}{39916800} + \ldots\right),$$

$$= x^2 + \sin x \sin t.$$

We arrive at the exact solution of the problem.

Solution of Example 1 by the DJM:

Consider Equation (27) with initial conditions: $u(x,0) = x^2$, $u_t(x,0) = \sin x$.

We integrate both sides of Equation (27) twice from 0 to t using the given initial condition and obtain

$$u(x,t) = -t^2 + x^2 + t\ \sin x + \int_0^t\int_0^t (\partial_{x,x}u)dsds, \tag{35}$$

By reducing the integration in Equation (35) from double to single (Wazwaz, 2015), we obtain

$$u(x,t) = -t^2 + x^2 + t\ \sin x + \int_0^t (t-s)(\partial_{xx}u)ds, \tag{36}$$

then,

$$u_0 = -t^2 + x^2 + t\ \sin x,$$

$$N(u_{n+1}) = \int_0^t (t-s)(\partial_{xx}u_n)ds,\ n \in \mathbb{N}\cup\{0\}.$$

The DJM algorithm gives the following iterations:

$$u_0 = -t^2 + x^2 + t\ \sin x,$$

$$u_1 = \int_0^t (t-s)(\partial_{xx}u_0)ds = t^2 - \frac{1}{6}t^3\sin x,$$

$$u_2 = \int_0^t (t-s)(\partial_{xx}(u_{0+}u_1))ds - u_1 = \frac{1}{120}t^5\sin x,$$

we find the rest of the iterations in the same way:

$$u_5 = -\frac{t^{11}\sin x}{39916800}$$

$$U_n = \sum_{i=0}^{n} u_i,\ n = 1,\ 2,\ \ldots$$

$$U_5 = u_1 + u_2 + u_3 + u_4 + u_5$$

$$U_5 = x^2 + t\ \sin x - \frac{1}{6}t^3\sin x + \frac{1}{120}t^5\sin x - \frac{t^7\sin x}{5040} + \frac{t^9\sin x}{362880}$$

is the same fifth iteration u_5 of the TAM solution. The exact solution can be obtained by

$$U(x,t) = \sum_{i=0}^{\infty} u_i = x^2 + \sin x \left(t \pm \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{t^7}{5040} + \frac{t^9}{362880} + \ldots\right),$$

$$= x^2 + \sin x \sin t.$$

Solving Example 1 by the BCM:

We consider Equation (27) with initial conditions: $u(x,0) = x^2$, $u_t(x,0) = \sin x$ and integrate both sides of Equation (27) twice from 0 to t using the given initial condition. We get

$$u(x,t) = -t^2 + x^2 + t\ \sin x + \int_0^t\int_0^t (\partial_{x,x}u)dsds. \tag{37}$$

Reducing the integration in Equation (37) from double to single (Wazwaz, 2015), we find

$$u(x,t) = -t^2 + x^2 + t\ \sin x + \int_0^t (t-s)(\partial_{xx}u)ds. \tag{38}$$

Let $u_0 = -t^2 + x^2 + t\ \sin x$ and $N(u_{n-1})$

$$= \int_0^t (t-s)(\partial_{xx}u_{n-1})ds,\ \ n \in \mathbb{N}. \tag{39}$$

Applying the BCM, we obtain:

$$u_0 = -t^2 + x^2 + t\ \sin\ x,$$

$$u_1 = u_0 + \int_0^t (t-s)(\partial_{xx}u_0)ds = x^2 + t\ \sin\ x - \frac{1}{6}t^3 \sin\ x,$$

$$u_2 = u_0 + \int_0^t (t-s)(\partial_{xx}u_1)ds = x^2 + t\sin\ x - \frac{1}{6}t^3 \sin\ x + \frac{1}{120}t^5 \sin\ x,$$

$$\vdots$$

$u_5 = x^2 + t\ \sin x - \frac{1}{6}t^3\sin x + \frac{1}{120}t^5\sin x - \frac{t^7 \sin x}{5040} + \frac{t^9 \sin x}{362880}$ is the same fifth iteration u_5 in the TAM.

The exact solution is obtained by taking a limit

$$u(x,t) = \lim_{n\to\infty} u_n.$$

$$u(x,t) = x^2 + \sin x\left(t + -\frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{t^7}{5040} + \frac{t^9}{362880} - \frac{t^{11}\sin x}{39916800} + \dots\right),$$
$$= x^2 + \sin x \sin t.$$

Example 2. Let us consider the 1D nonlinear wave equation (Wazwaz, 2007)

$$u_{tt} = u_{xx} + u + u^2 - xt - x^2t^2, \tag{40}$$

with initial conditions:

$$u(x,0) = 0, u_t(x,0) = x.$$

In order to solve Equation (40) by TAM with the initial conditions given, we have the following form

$$\begin{aligned} L(u) &= u_{tt}(x,t), \\ N(u) &= u_{xx}(x,t) + u(x,t) + u(x,t)^2, \\ g(x,t) &= -xt - x^2t^2, \end{aligned} \tag{41}$$

The initial problem is

$$L(u_0) = -xt - x^2t^2 \text{ with } u_0(x,0) = 0, u_{0t}(x,0) = x. \tag{42}$$

We make use of the generalized iterative formula

$$L(u_{n+1}) + N(u_n) = g(x,t),\ \ u_{n+1}(x,0) = 0, u_{(n+1)t}(x,0) = x.$$

By solving Equation (42) we get

$$u_0 = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12},$$

The first iteration $u_1(x,t)$ can be evaluated by solving

$$u_{1tt} = u_{0xx}(x,t) + u_0(x,t) + u_0(x,t)^2 - xt - x^2t^2,$$

with $u_1(x,0) = 0, u_{1t}(x,0) = x$.

The solution is

$$u_1 = -\frac{t^6}{180} + tx - \frac{t^5x}{120} - \frac{t^6x^2}{72} + \frac{t^8x^2}{2016} - \frac{t^7x^3}{252} + \frac{t^9x^3}{2592} + \frac{t^{10}x^4}{12960},$$

Applying the same process for u_2, we have

$$u_{2tt} = u_{1xx}(x,t) + u_1(x,t) + u_1(x,t)^2 - xt - x^2t^2 \text{ with } u_2(x,0) = 0, u_{2t}(x,0) = x.$$

By solving this problem, we get

$$u_2 = -\frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^{14}}{5896800} + tx - \frac{t^7x}{5040} - \frac{11t^9x}{22680} + \frac{t^{11}x}{47520} + \frac{t^{13}x}{1684800} - \frac{11t^8x^2}{20160} + \frac{t^{10}x^2}{181440} + \dots.$$

$$\vdots$$

$$\begin{aligned} u_5 = &-\frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^{14}}{5896800} + tx - \frac{t^7x}{5040} - \frac{11t^9x}{22680} \\ &+ \frac{t^{11}x}{47520} + \frac{t^{13}x}{1684800} - \frac{11t^8x^2}{20160} + \frac{t^{10}x^2}{181440} + \frac{43t^{12}x^2}{5702400} \\ &+ \frac{t^{14}x^2}{1179360} - \frac{t^{16}x^2}{43545600} - \frac{t^9x^3}{2268} + \frac{5t^{11}x^3}{399168} + \frac{t^{13}x^3}{673920} \\ &+ \frac{13t^{15}x^3}{76204800} - \frac{t^{17}x^3}{63452160} - \frac{t^{10}x^4}{11340} + \frac{t^{12}x^4}{155520} + \\ &\vdots \end{aligned} \tag{43}$$

This series converges to the exact solution when

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \ xt.$$

Solving Example 2 by the DJM:

Consider Equation (40) with the initial conditions $u(x,0) = 0, u_t(x,0) = x$. Integrating both sides of Equation (40) twice from 0 to t, we get

$$u(x,t) = \ tx - \frac{t^3x}{6} - \frac{t^4x^2}{12} + \int_0^t\int_0^t \partial_{x,x}u + u + u^2 dsds, \tag{44}$$

and reducing the integration in Equation (44) from double to single (Wazwaz, 2015), we find

$$u(x,t) = \ t^2 - \frac{t^4}{6} + x^2 - t^2x^2 + \int_0^t (t-s)(\partial_{x,x}u + u + u^2)ds. \tag{45}$$

Therefore, we have the following recurrence relation

$$u_0 = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12},$$

$$N(u_{n+1}) = \int_0^t (t-s)(\partial_{x,x}u_n + u_n + {u_n}^2)ds,\ \ n \in \mathbb{N}\cup\{0\}.$$

By applying the DJM, we find

$$
\begin{aligned}
u_0 &= tx - \frac{t^3x}{6} - \frac{t^4x^2}{12},\\
u_1 &= -\frac{t^6}{180} + \frac{t^3x}{6} - \frac{t^5x}{120} + \frac{t^4x^2}{12} - \frac{t^6x^2}{72} + \frac{t^8x^2}{2016}\\
&\quad - \frac{t^7x^3}{252} + \frac{t^9x^3}{2592} + \frac{t^{10}x^4}{12960},\\
u_2 &= \frac{t^6}{180} - \frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^{14}}{5896800} + \frac{t^5x}{120}\\
&\quad - \frac{t^7x}{5040} - \frac{11t^9x}{22680} + \frac{t^{11}x}{47520} + \frac{t^{13}x}{1684800}\\
&\quad + \frac{t^6x^2}{72} - \frac{t^8x^2}{960} + \frac{t^{10}x^2}{181440} + \cdots\\
&\ \vdots\\
u_5 &= \frac{t^{12}}{3326400} + \frac{239t^{14}}{1452971520} - \frac{6619t^{16}}{435891456000}\\
&\quad - \frac{29839t^{18}}{72754246656000} + \frac{60709t^{20}}{6911653432320000}\\
&\quad - \frac{68107471t^{22}}{46833363657400320000} + \ldots.
\end{aligned}
$$

$$U_n = \sum_{i=0}^{n} u_i \quad n = 1,\ 2,\ 3, \ldots.$$

$$
\begin{aligned}
U_5 &= u_1 + u_2 + u_3 + u_4 + u_5,\\
U_5 &= -\frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^{14}}{5896800} + tx - \frac{t^7x}{5040}\\
&\quad - \frac{11t^9x}{22680} + \frac{t^{11}x}{47520} + \frac{t^{13}x}{1684800} - \frac{11t^8x^2}{20160}\\
&\quad + \frac{t^{10}x^2}{181440} + \frac{43t^{12}x^2}{5702400} + \frac{t^{14}x^2}{1179360}\\
&\quad - \frac{t^{16}x^2}{43545600} - \frac{t^9x^3}{2268} + \frac{5t^{11}x^3}{399168} + \frac{t^{13}x^3}{673920}\\
&\quad + \frac{13t^{15}x^3}{76204800} - \frac{t^{17}x^3}{63452160} - \frac{t^{10}x^4}{11340} + \frac{t^{12}x^4}{155520} + \ldots.
\end{aligned}
$$

This is the same as the approximate solution in Equation (43) which converges to the exact solution

$$U(x,t) = \sum_{i=0}^{\infty} u_i = xt.$$

Solving Example 2 by the BCM:
Consider Equation (43) by following the same way as in the DJM, we get Equation (45) So, let

$$u_0 = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12},$$
$$N(u_{n-1}) = \int_0^t (t-s)(\partial_{x,x}u_{n-1} + u_{n-1} + u_{n-1}{}^2)ds,\ \ n \in \mathbb{N}.$$

By applying the BCM, we obtain:

$$
\begin{aligned}
u_0 &= tx - \frac{t^3x}{6} - \frac{t^4x^2}{12},\\
u_1 &= -\frac{t^6}{180} + tx - \frac{t^5x}{120} - \frac{t^6x^2}{72} + \frac{t^8x^2}{2016} - \frac{t^7x^3}{252} + \frac{t^9x^3}{2592} + \frac{t^{10}x^4}{12960},\\
u_2 &= -\frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^{14}}{5896800} + tx - \frac{t^7x}{5040}\\
&\quad - \frac{11t^9x}{22680} + \frac{t^{11}x}{47520} + \frac{t^{13}x}{1684800} - \frac{11t^8x^2}{20160}\\
&\quad + \frac{t^{10}x^2}{181440} + \ldots.\\
&\ \vdots\\
u_5 &= -\frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^{14}}{5896800} + tx - \frac{t^7x}{5040} - \frac{11t^9x}{22680}\\
&\quad + \frac{t^{11}x}{47520} + \frac{t^{13}x}{1684800} - \frac{11t^8x^2}{20160} + \frac{t^{10}x^2}{181440}\\
&\quad + \frac{43t^{12}x^2}{5702400} + \frac{t^{14}x^2}{1179360} - \frac{t^{16}x^2}{43545600} - \frac{t^9x^3}{2268}\\
&\quad + \frac{5t^{11}x^3}{399168} + \frac{t^{13}x^3}{673920} + \frac{13t^{15}x^3}{76204800} - \frac{t^{17}x^3}{63452160}\\
&\quad - \frac{t^{10}x^4}{11340} + \frac{t^{12}x^4}{155520} + \ldots
\end{aligned}
$$

is the same of the approximate solution in Equation (43) We see that the approximate solutions obtained from the three proposed techniques are the same.

To prove the convergence analysis for the proposed methods, we will use the process given in Equations(21)–(24). The iterative scheme for Equation (43) can be formulated as

$$v_0(x,t) = u_0(x,t) = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12}.$$

Applying the TAM, the operator $F[v_k]$ as defined in Equation (22) with the term S_k which is the solution for the following problem, it will be then

$$
\begin{aligned}
v_{ktt}(x,t) &= \left(\sum\nolimits_{i=0}^{k-1} v_{ixx}(x,t)\right) + \left(\sum\nolimits_{i=0}^{k-1} v_i(x,t)\right)\\
&\quad + \left(\sum\nolimits_{i=0}^{k-1} v_i(x,t)\right)^2 - xt - x^2t^2,
\end{aligned}
$$

with

$$v_k(x,0) = 0,\ \ v_{kt}(x,0) = x\ \ k \geq 1.$$

Also, when applying the BCM, the S_k represents the solution for the following problem,

$$
\begin{aligned}
v_k &= v_0 + \left(\sum\nolimits_{i=0}^{k-1} v_{ixx}(x,t)\right) + \left(\sum\nolimits_{i=0}^{k-1} v_i(x,t)\right)\\
&\quad + \left(\sum\nolimits_{i=0}^{k-1} v_i(x,t)\right)^2, k \geq 1.
\end{aligned}
$$

Iterative approximations can be used directly when applying the DJM. Therefore, we have the following terms

$$v_1 = -\frac{t^6}{180}+\frac{t^3x}{6}-\frac{t^5x}{120}+\frac{t^4x^2}{12}-\frac{t^6x^2}{72}+\frac{t^8x^2}{2016} -\frac{t^7x^3}{252}+\frac{t^9x^3}{2592}+\frac{t^{10}x^4}{12960},$$

$$v_2 = \frac{t^6}{180}-\frac{t^8}{1680}+\frac{t^{10}}{90720}+\frac{t^{14}}{5896800}+\frac{t^5x}{120} -\frac{t^7x}{5040}-\frac{11t^9x}{22680}+\dots,$$

$$v_3 = \frac{t^8}{1680}-\frac{t^{10}}{33600}+\frac{t^{12}}{5987520}-\frac{t^{14}}{11531520} +\frac{11t^{16}}{1415232000}+\frac{47t^{18}}{46637337600} -\frac{t^{20}}{28957824000}+\dots.$$

We use the above duplicates in computing the values of β_i for the equation as in Equation (26) we obtain

$$\beta_0 = \frac{||v_1||}{||v_0||} = 0.417171<1,$$
$$\beta_1 = \frac{||v_2||}{||v_1||} = 0.190979<1,$$
$$\beta_2 = \frac{||v_3||}{||v_2||} = 0.114691<1,$$
$$\beta_3 = \frac{||v_4||}{||v_3||} = 0.0696193<1,$$
$$\beta_4 = \frac{||v_5||}{||v_4||} = 0.0458687<1,$$

where, the β_i values for $i \geq 0$ and $\forall(x,t): \; x\in\mathbb{R}$, $0<x\leq 1$ are less than 1, so the proposed iterative methods satisfy the convergence.

We calculate the absolute error $Absr_n = N[Abs[w-u_n]]$, to check the accuracy of the approximate solution (u_n), where $w = xt$ is the exact solution. Figures 1 and 2 show the 3D plotted graph of the $Absr_n$, for the approximate solution obtained by the suggested iterative methods and BDIM. The results show that BDIM accuracy grows with shortening of the time step. This kind of behaviour is expected, since shorter time step enables better time resolution and captures the solution development more accurately. Similarly, by increasing the number iterations for iterative methods, the errors are decreasing and the precision of the approximate solution increases.

Example 3. Consider 2D linear wave equation given in equation (Qin, 2009)

$$u_{tt} - \left(au_{xx} + bu_{yy}\right) = 0, \tag{46}$$

with the initial conditions : $u(x,y,0) = e^{x+y}$, $u_t(x,y,0) = -\sqrt{2}\, e^{x+y}$.

Where $a = 1$ and $b = 1$ (Qin, 2009), Equation (46) will be solved by the three proposed iterative methods.

Solving Example 3 by the TAM:

By applying the TAM, we obtain the following iterations

$$u_0 = e^{x+y} - \sqrt{2}e^{x+y}t.$$

$$u_1 = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3.$$

$$u_2 = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}},$$

$$\vdots$$

$$\begin{aligned} u_5 = {} & e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 \\ & + \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}} + \frac{1}{90}e^{x+y}t^6 - \frac{e^{x+y}t^7}{315\sqrt{2}} \\ & + \frac{e^{x+y}t^8}{2520} - \frac{e^{x+y}t^9}{11340\sqrt{2}} + \frac{e^{x+y}t^{10}}{113400} \\ & - \frac{e^{x+y}t^{11}}{623700\sqrt{2}}, \end{aligned} \tag{47}$$

$$u(x,y,t) = \lim_{n\to\infty} u_n = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \dots.$$

Solving Example 3 by the DJM:

Consider Equation (46) with initial conditions : $u(x,y,0) = e^{x+y}$, $u_t(x,y,0) = -\sqrt{2}e^{x+y}$ Log (e).

By applying the DJM, we get

$$u_0 = e^{x+y} - \sqrt{2}e^{x+y}t,$$

$$u_1 = e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3,$$

$$u_2 = \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}},$$

$$\vdots$$

$$u_5 = \frac{e^{x+y}t^{10}}{113400} - \frac{e^{x+y}t^{11}}{623700\sqrt{2}},$$

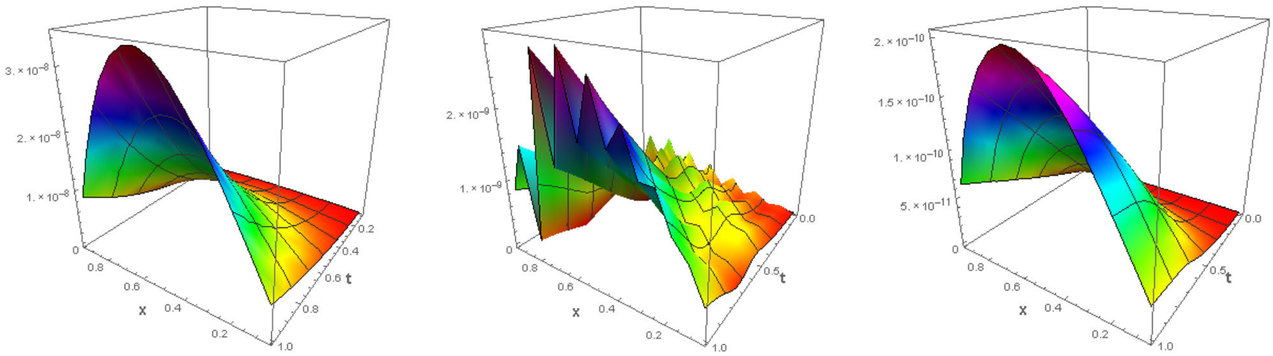
$$U_n = \sum_{i=0}^{n} u_i \quad n = 1,\; 2,\; \dots.$$

$$\begin{aligned} U_5 &= u_0 + u_1 + u_2 + u_3 + u_4 + u_5 \\ &= e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 \\ &\quad - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}} \\ &\quad + \frac{1}{90}e^{x+y}t^6 - \frac{e^{x+y}t^7}{315\sqrt{2}} + \frac{e^{x+y}t^8}{2520} \\ &\quad - \frac{e^{x+y}t^9}{11340\sqrt{2}} + \frac{e^{x+y}t^{10}}{113400} - \frac{e^{x+y}t^{11}}{623700\sqrt{2}} \end{aligned}$$

is the same as the solution in Equation (47) we can get the exact solution by

$$U(x,y,t) = \sum_{i=0}^{\infty} u_i = e^{x+y-\sqrt{2}t}.$$

Solving Example 3 by the BCM:



(a) $\Delta t = 0.1$ (b) $\Delta t = 0.01$ (c) $\Delta t = 0.001$

Figure 1. Absolute errors obtained by the BDIM method using 17 equidistant nodes and three different time steps (0.1, 0.01 and 0.001) for text Example 2 are shown. In all three cases, we observe that the error decreases with time.

Consider Equation (46) with initial conditions $u(x,y,0)=e^{x+y}$, $u_t(x,y,0)=-\sqrt{2}e^{x+y}$ Log (e).

Applying the BCM, we obtain:

$$u_0 = e^{x+y} - \sqrt{2}e^{x+y}t,$$

$$u_1 = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3,$$

$$u_2 = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}},$$

$$\vdots$$

$$u_5 = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}} + \frac{1}{90}e^{x+y}t^6 - \frac{e^{x+y}t^7}{315\sqrt{2}} + \frac{e^{x+y}t^8}{2520} - \frac{e^{x+y}t^9}{11340\sqrt{2}} + \frac{e^{x+y}t^{10}}{113400} - \frac{e^{x+y}t^{11}}{623700\sqrt{2}},$$

$$u(x,y,t) = \lim_{n\to\infty} u_n = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \ldots.$$

This series converges to the exact solution

$$u(x,y,t) = e^{x+y-\sqrt{2}t} = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\left(\sqrt{2}e^{x+y}\right)t^3 + \ldots.$$

Also, is the same as the solution in Equation (47) and the exact solution can be obtained by $u(x,y,t) = \lim_{n\to\infty} u_n = e^{x+y-\sqrt{2}t}$.

Example 4. We take the following 2D nonlinear wave equations

$$u_{tt}(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) - u(x,y,t)^2 + t^2x^2y^2, \tag{48}$$

with initial conditions:

$$u(x,y,0) = 0, \quad u_t(x,y,0) = xy.$$

Equation (48) will be solved by the three iterative methods with the initial conditions.

Solving Example 4 by the TAM:

$$u_0 = txy + \frac{1}{12}t^4x^2y^2,$$

$$u_1 = \frac{t^6x^2}{180} + txy + \frac{t^6y^2}{180} - \frac{1}{252}t^7x^3y^3 - \frac{t^{10}x^4y^4}{12960},$$

$$u_2 = \frac{t^8}{2520} - \frac{t^{14}x^4}{5896800} + txy - \frac{11t^9x^3y}{22680} - \frac{t^{14}x^2y^2}{2948400} - \frac{t^{12}x^4y^2}{142560} - \frac{11t^9xy^3}{22680} + \frac{t^{15}x^5y^3}{4762800} - \frac{t^{14}y^4}{5896800} - \frac{t^{12}x^2y^4}{142560} + \ldots,$$

continuing in this way till $n = 4$, we find

$$u_4 = \frac{t^8}{2520} - \frac{t^{14}x^4}{5896800} + txy - \frac{11t^9x^3y}{22680} - \frac{t^{14}x^2y^2}{2948400} - \frac{t^{12}x^4y^2}{142560} - \frac{11t^9xy^3}{22680} + \frac{t^{15}x^5y^3}{4762800} - \frac{t^{14}y^4}{5896800} - \frac{t^{12}x^2y^4}{142560} + \frac{t^{10}x^4y^4}{11340} + \frac{t^{18}x^6y^4}{356918400} + \frac{t^{15}x^3y^5}{4762800} + \frac{t^{13}x^5y^5}{1010880} + \frac{t^{18}x^4y^6}{356918400} - \frac{t^{16}x^6y^6}{15240960} - \frac{t^{19}x^7y^7}{558472320} - \frac{t^{22}x^8y^8}{77598259200} + \ldots \tag{49}$$

This series converges to the exact solution when $u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t) = x\ y\ t.$

Solving Example 4 by the DJM:

$$u_0 = txy + \frac{1}{12}t^4x^2y^2,$$

$$u_1 = \frac{t^6x^2}{180} + \frac{t^6y^2}{180} - \frac{1}{12}t^4x^2y^2 - \frac{1}{252}t^7x^3y^3 - \frac{t^{10}x^4y^4}{12960},$$

$$u_2 = \frac{t^8}{2520} - \frac{t^6x^2}{180} - \frac{t^{14}x^4}{5896800} - \frac{11t^9x^3y}{22680} - \frac{t^6y^2}{180} - \frac{t^{14}x^2y^2}{2948400} - \frac{t^{12}x^4y^2}{142560} - \frac{11t^9xy^3}{22680} + \frac{1}{252}t^7x^3y^3 + \ldots.$$

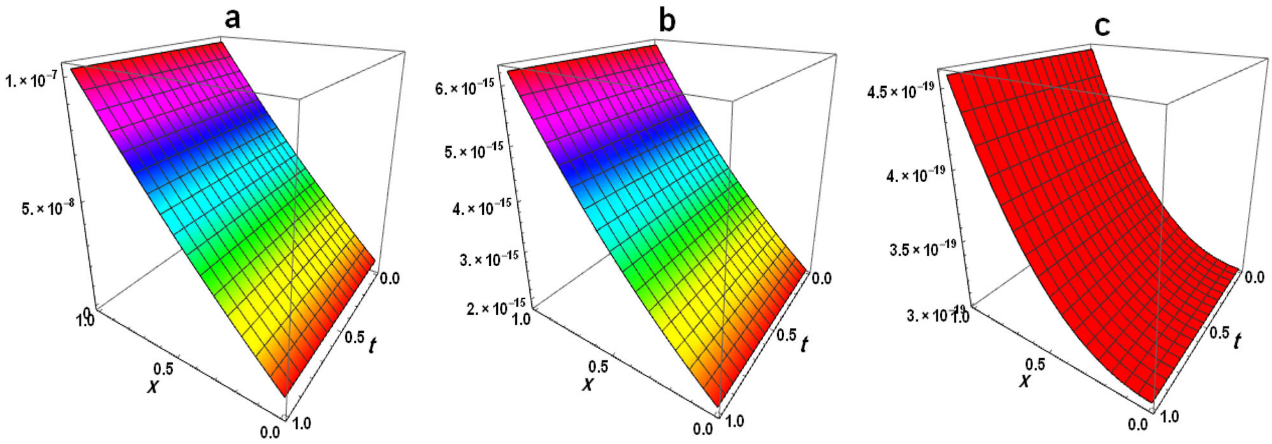


Figure 2. We present the absolute error $Absr_n$ versus time and x for test Example 2 at $n = 1,\ 3,\ 4.$ Of the panes present $u_1(x,t)(a), u_3(x,t)(b)$ and $u_4(x,t)\ (c)$, at the time instant $t = 0.1$. Very good accuracy increase is observed when the number of iterations n is increasing.

Continue to till $n = 4$

$$u_4 = \frac{139t^{18}}{378928368000} - \frac{t^{38}}{530921529398163456000} - \frac{t^{16}x^2}{3891888000} + \frac{t^{26}x^2}{166617032400000} + \ldots.$$

$$U_n = \sum_{i=0}^{n} u_i \quad n = 1,\ 2,\ \ldots.$$

$$U_4 = u_1 + u_2 + u_3 + u_4.$$

$$\begin{aligned}U_4 = {} & \frac{t^8}{2520} - \frac{t^{14}x^4}{5896800} + txy - \frac{11t^9x^3y}{22680} - \frac{t^{14}x^2y^2}{2948400} \\ & - \frac{t^{12}x^4y^2}{142560} - \frac{11t^9xy^3}{22680} + \frac{t^{15}x^5y^3}{4762800} - \frac{t^{14}y^4}{5896800} \\ & - \frac{t^{12}x^2y^4}{142560} + \frac{t^{10}x^4y^4}{11340} + \frac{t^{18}x^6y^4}{356918400} + \frac{t^{15}x^3y^5}{4762800} \\ & + \frac{t^{13}x^5y^5}{1010880} + \frac{t^{18}x^4y^6}{356918400} - \frac{t^{16}x^6y^6}{15240960} \\ & - \frac{t^{19}x^7y^7}{558472320} - \frac{t^{22}x^8y^8}{77598259200} + \ldots.\end{aligned}$$

This is the same as the approximate solution in Equation (49) and converges to the exact solution

$$U = \sum_{i=0}^{\infty} u_i = xyt.$$

Solving Example 4 by the BCM:

$$u_0 = txy + \frac{1}{12}t^4x^2y^2,$$

$$u_1 = \frac{t^6x^2}{180} + txy + \frac{t^6y^2}{180} - \frac{1}{252}t^7x^3y^3 - \frac{t^{10}x^4y^4}{12960},$$

$$\begin{aligned}u_2 = {} & \frac{t^8}{2520} - \frac{t^{14}x^4}{5896800} + txy - \frac{11t^9x^3y}{22680} - \frac{t^{14}x^2y^2}{2948400} \\ & - \frac{t^{12}x^4y^2}{142560} - \frac{11t^9xy^3}{22680} + \frac{t^{15}x^5y^3}{4762800} - \frac{t^{14}y^4}{5896800} \\ & - \frac{t^{12}x^2y^4}{142560} + \ldots,\end{aligned}$$

$$\vdots$$

$$\begin{aligned}u_4 = {} & \frac{t^8}{2520} - \frac{t^{14}x^4}{5896800} + txy - \frac{11t^9x^3y}{22680} - \frac{t^{14}x^2y^2}{2948400} \\ & - \frac{t^{12}x^4y^2}{142560} - \frac{11t^9xy^3}{22680} + \frac{t^{15}x^5y^3}{4762800} - \frac{t^{14}y^4}{5896800} \\ & - \frac{t^{12}x^2y^4}{142560} + \frac{t^{10}x^4y^4}{11340} + \frac{t^{18}x^6y^4}{356918400} + \frac{t^{15}x^3y^5}{4762800} \\ & + \frac{t^{13}x^5y^5}{1010880} + \frac{t^{18}x^4y^6}{356918400} - \frac{t^{16}x^6y^6}{15240960} \\ & - \frac{t^{19}x^7y^7}{558472320} - \frac{t^{22}x^8y^8}{77598259200} + \ldots\end{aligned}$$

is the same approximate solution as in Equation (49).

To prove the convergence analysis for the proposed methods, we can find the β_i values for the problem as in Equation (48) Hence, the terms of the series $\sum_{i=0}^{\infty} v_i(x,y,t)$ given in Equation (24) we have

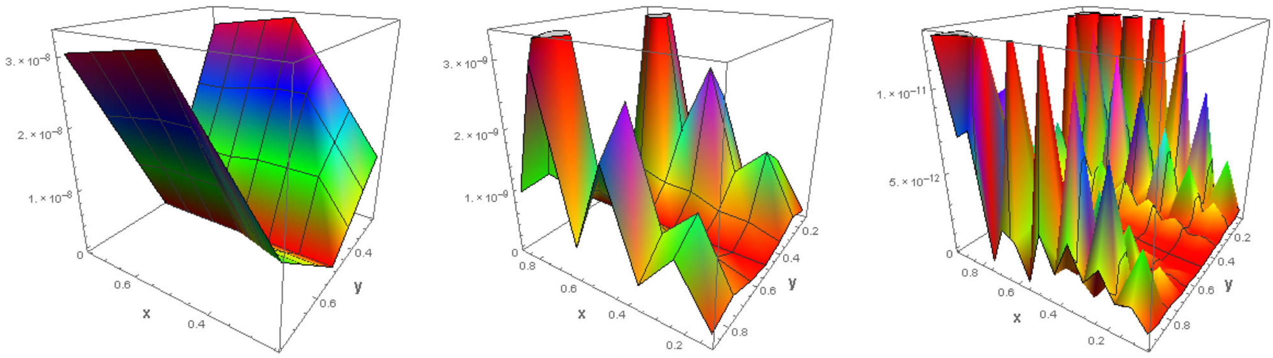
$$\beta_0 = \frac{||v_1||}{||v_0||} = 0.0863236 < 1,$$

$$\beta_1 = \frac{||v_2||}{||v_1||} = 0.437467 < 1,$$

$$\beta_2 = \frac{||v_3||}{||v_2||} = 0.110856 < 1,$$

$$\beta_3 = \frac{||v_4||}{||v_3||} = 0.279038 < 1,$$

where, the β_i values for $i \geq 0$ and $\forall (x,y) \in \mathbb{R}^2,\ 0 < x,y,t \leq 1$ are less than 1, so the proposed iterative methods satisfy the convergence. In order to test the accuracy of the approximate solution, we calculate the $Absr_n$ where $w = xyt$ is the exact solution. Figures 3 and 4 show the absolute error $Absr_n$ for approximate solutions obtained by the iterative methods and BDIM. It can be seen clearly that by increasing the number of iterations the error of iterative methods is reduced and the solution becomes more accurate. The same conclusion can be drawn



(a)5^2 nodes (b)9^2 nodes (c)17^2 nodes

Figure 3. The panels show absolute errors obtained by the BDIM solution of Example 4 using a time step of $\Delta t = 0.01$ for three different mesh discretizations – 5^2 nodes (a), 9^2 nodes (b) and 17^2 nodes(c). We observe a substantial improvement in results accuracy, when a more fine computational grid is used. Results are shown at $t = 0.1$.

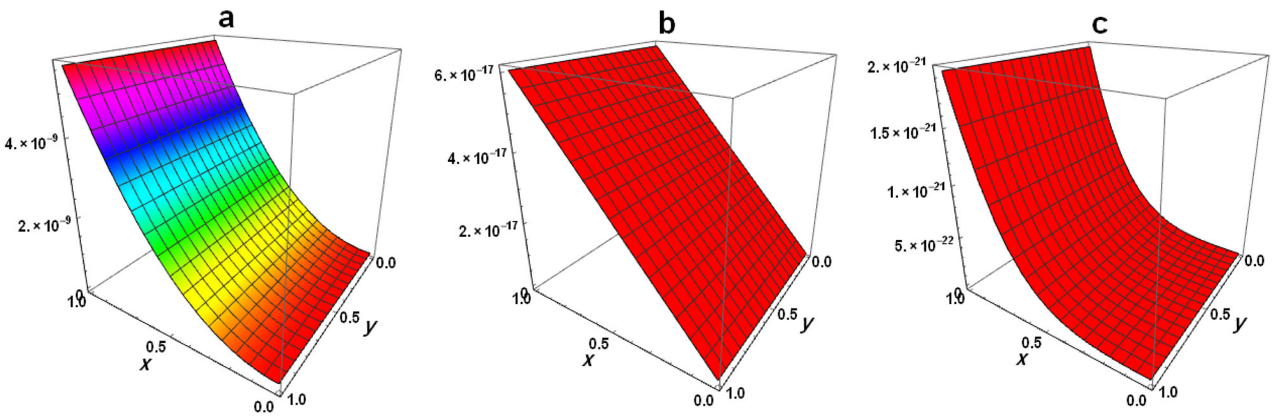


Figure 4. *(a–c)*:The absolute error $Absr_n$ of the solution of Example 4 at $t = 0.1$ for $n = 1,\ 3,\ 4.$ The panels show $u_1(x, y, t)(a)$, $u_3(x, y, t)(b)$ and $u_4(x, y, t)\ (c)$. We observe an increase of accuracy as the number of iterations n increases.

for BDIM, when computational mesh density is increased (Figure 3).

Example 5. Let us take the following 3D linear wave equation given as (Wazwaz, 2010).

$$u_{tt} = u_{xx} + u_{yy} + u_{zz} + \sin x + \sin y, \qquad (50)$$

with initial conditions:

$$u(x, y, z, 0) = \sin x + \sin y, u_t(x, y, z, 0) = \sin z.$$

Equation (50) will be solved by the three proposed iterative methods

Solving Example 5 by the TAM:

$$u_0 = \sin x + \frac{1}{2}t^2\sin x + \sin y + \frac{1}{2}t^2\sin y + t\sin z.$$

$$u_1 = \sin x - \frac{1}{24}t^4\sin x + \sin y - \frac{1}{24}t^4\sin y + t\sin z - \frac{1}{6}t^3\sin z,$$

$$u_2 = \sin x + \frac{1}{720}t^6\sin x + \sin y + \frac{1}{720}t^6\sin y + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z,$$

$$\vdots$$

$$u_5 = \sin x - \frac{t^{12}\sin x}{479001600} + \sin y - \frac{t^{12}\sin y}{479001600} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800}. \qquad (51)$$

$$\begin{aligned} u(x, y, z, t) &= \lim_{n\to\infty} u_n \\ &= \sin x - \frac{t^{16}\sin x}{20922789888000} + \sin y - \frac{t^{16}\sin y}{20922789888000} + t\sin z - \frac{1}{6}t^3\sin z \\ &\quad + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800} + \frac{t^{13}\sin z}{6227020800} \\ &\quad - \frac{t^{15}\sin z}{1307674368000} + \dots \\ &= \sin x + \sin y + \sin z \sin t \end{aligned}$$

is the exact solution.

Solving Example 5 by the DJM:

$$u_0 = \sin x + \frac{1}{2}t^2\sin x + \sin y + \frac{1}{2}t^2\sin y + t\sin z,$$

$$u_1 = -\frac{1}{2}t^2\mathrm{sinx} - \frac{1}{24}t^4\mathrm{sinx} - \frac{1}{2}t^2\mathrm{siny} - \frac{1}{24}t^4\mathrm{siny} - \frac{1}{6}t^3\mathrm{sinz},$$

$$u_2 = \frac{1}{24}t^4\mathrm{sinx} + \frac{1}{720}t^6\mathrm{sinx} + \frac{1}{24}t^4\mathrm{siny} + \frac{1}{720}t^6\mathrm{siny} + \frac{1}{120}t^5\mathrm{sinz},$$

$$\vdots$$

$$u_5 = -\frac{t^{10}\mathrm{sinx}}{3628800} - \frac{t^{12}\mathrm{sinx}}{479001600} - \frac{t^{10}\mathrm{siny}}{3628800} - \frac{t^{12}\mathrm{siny}}{479001600} - \frac{t^{11}\mathrm{sinz}}{39916800}.$$

$$U_n = \sum_{i=0}^{n} u_i n = 1,\ 2,\ 3, 4, \ldots$$

$$U_5 = u_1 + u_2 + u_3 + u_4 + u_5.$$

$$U_5 = \mathrm{sinx} - \frac{t^{12}\mathrm{sinx}}{479001600} + \mathrm{siny} - \frac{t^{12}\mathrm{siny}}{479001600} + t\mathrm{sinz} - \frac{1}{6}t^3\mathrm{sinz} + \frac{1}{120}t^5\mathrm{sinz} - \frac{t^7\mathrm{sinz}}{5040} + \frac{t^9\mathrm{sinz}}{362880} - \frac{t^{11}\mathrm{sinz}}{39916800},$$

is the same as the solution in Equation (51), the exact solution can be obtained by

$$U = \sum_{i=0}^{\infty} u_i = \sin x - \frac{t^{16}\sin x}{20922789888000} + \sin y - \frac{t^{16}\sin y}{20922789888000} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800} + \frac{t^{13}\sin z}{6227020800} - \frac{t^{15}\sin z}{1307674368000} + \cdots.$$
$$= \sin x + \sin y + \sin z \sin t.$$

Solving Example 5 by the BCM:

$$u_0 = \sin x + \frac{1}{2}t^2\sin x + \sin x + \frac{1}{2}t^2\sin x + t\sin x,$$

$$u_1 = \sin x - \frac{1}{24}t^4\sin x + \sin y - \frac{1}{24}t^4\sin y + t\sin z - \frac{1}{6}t^3\sin z,$$

$$u_2 = \sin x + \frac{1}{720}t^6\sin x + \sin y + \frac{1}{720}t^6\sin y + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z,$$

$$\vdots$$

$$u_5 = \sin x - \frac{t^{12}\sin x}{479001600} + \sin y - \frac{t^{12}\sin y}{479001600} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800},$$

is the same as the solution in Equation (51), the exact solution can be found by

$$u(x,y,z,t) = \lim_{n\to\infty} u_n$$
$$= \sin x - \frac{t^{16}\sin x}{20922789888000} + \sin y - \frac{t^{16}\sin y}{20922789888000} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800} + \frac{t^{13}\sin z}{6227020800} - \frac{t^{15}\sin z}{1307674368000} + \ldots,$$
$$= \sin x + \sin y + \sin z \sin t.$$

Example 6. Consider 3D nonlinear wave equation given in equation

$$u_{tt}(x,y,z,t) = u_{xx}(x,y,z,t) + u_{yy}(x,y,z,t) + u_{zz}(x,y,z,t) - u(x,y,z,t)^2 + t^2x^2y^2z^2, \tag{52}$$

with the initial conditions:

$$u(x,y,z,0) = 0,\ \ u_t(x,y,z,0) = xyz.$$

Solving Example 6 by the TAM:

$$u_0 = txyz + \frac{1}{12}t^4x^2y^2z^2,$$

$$u_1 = \frac{1}{180}t^6x^2y^2 + txyz + \frac{1}{180}t^6x^2z^2 + \frac{1}{180}t^6y^2z^2 - \frac{1}{252}t^7x^3y^3z^3 - \frac{t^{10}x^4y^4z^4}{12960},$$

$$u_2 = \frac{t^8x^2}{2520} + \frac{t^8y^2}{2520} - \frac{t^{14}x^4y^4}{5896800} + txyz - \frac{11t^9x^3y^3z}{22680} + \frac{t^8z^2}{2520} - \frac{t^{14}x^4y^2z^2}{2948400} - \frac{t^{14}x^2y^4z^2}{2948400} - \frac{t^{12}x^4y^4z^2}{142560} - \frac{11t^9x^3yz^3}{22680} - \frac{11t^9xy^3z^3}{22680}$$

$$\vdots$$

$$u_4 = \frac{t^{10}}{37800} - \frac{t^{18}x^4}{1943222400} - \frac{t^{18}x^2y^2}{971611200} - \frac{t^{16}x^4y^2}{88452000} - \frac{t^{18}y^4}{1943222400} - \frac{t^{16}x^2y^4}{88452000} - \frac{t^{14}x^4y^4}{12972960} + \frac{t^{24}x^6y^4}{4101342336000} + \ldots. \tag{53}$$

This is the approximate solution, which converges to the exact solution when

$$u(x,y,z,t) = \lim_{n\to\infty} u_n = xyzt.$$

Solving Example 6 by the DJM:

$$u_0 = txyz + \frac{1}{12}t^4x^2y^2z^2,$$

$$u_1 = \frac{1}{180}t^6x^2y^2 + \frac{1}{180}t^6x^2z^2 + \frac{1}{180}t^6y^2z^2 - \frac{1}{12}t^4x^2y^2z^2 - \frac{1}{252}t^7x^3y^3z^3 - \frac{t^{10}x^4y^4z^4}{12960},$$

$$u_2 = \frac{t^8x^2}{2520} + \frac{t^8y^2}{2520} - \frac{1}{180}t^6x^2y^2 - \frac{t^{14}x^4y^4}{5896800} - \frac{11t^9x^3y^3z}{22680} + \frac{t^8z^2}{2520} - \frac{1}{180}t^6x^2z^2 + \ldots.$$

$$\vdots$$

$$u_4 = -\frac{t^{10}}{37800} - \frac{t^{22}}{660124080000} - \frac{t^{20}x^2}{36921225600} + \frac{139t^{18}x^4}{378928368000} + \frac{t^{30}x^4}{31952405923200000} - \frac{t^{38}x^8}{530921529398163456000} - \frac{t^{20}y^2}{36921225600} - \frac{t^{18}x^2y^2}{5262894000} + \ldots$$

$$U_n = \sum_{i=0}^{n} u_i \quad n = 1,\ 2,\ 3,\ldots.$$

$$U_4 = u_1 + u_2 + u_3 + u_4 = \frac{t^{10}}{37800} - \frac{t^{18}x^4}{1943222400} - \frac{t^{18}x^2y^2}{971611200} - \frac{t^{16}x^4y^2}{88452000} - \frac{t^{18}y^4}{1943222400} - \frac{t^{16}x^2y^4}{88452000} - \frac{t^{14}x^4y^4}{12972960} + \frac{t^{24}x^6y^4}{4101342336000} + \ldots.$$

This is the same as the approximate solution in Equation (53) and converges to the exact solution when

$$U = \sum_{i=0}^{\infty} u_i = xyzt.$$

Solving Example 6 by the BCM:

$$u_0 = txyz + \frac{1}{12}t^4x^2y^2z^2,$$

$$u_1 = \frac{1}{180}t^6x^2y^2 + txyz + \frac{1}{180}t^6x^2z^2 + \frac{1}{180}t^6y^2z^2 - \frac{1}{252}t^7x^3y^3z^3 - \frac{t^{10}x^4y^4z^4}{12960},$$

$$u_2 = \frac{t^8x^2}{2520} + \frac{t^8y^2}{2520} - \frac{t^{14}x^4y^4}{5896800} + txyz - \frac{11t^9x^3y^3z}{22680} + \frac{t^8z^2}{2520} - \frac{t^{14}x^4y^2z^2}{2948400} - \frac{t^{14}x^2y^4z^2}{2948400} - \frac{t^{12}x^4y^4z^2}{142560} - \frac{11t^9x^3yz^3}{22680} - \frac{11t^9xy^3z^3}{22680} + \ldots.$$

$$u_4 = \frac{t^{10}}{37800} - \frac{t^{18}x^4}{1943222400} - \frac{t^{18}x^2y^2}{971611200} - \frac{t^{16}x^4y^2}{88452000} - \frac{t^{18}y^4}{1943222400} - \frac{t^{16}x^2y^4}{88452000} - \frac{t^{14}x^4y^4}{12972960} + \frac{t^{24}x^6y^4}{4101342336000} + \ldots.$$

This is the same as the approximate solution in Equation (53) and converges to the exact solution.

To prove the state of convergence we find values of β_i for the problem . Hence, the terms of the series $\sum_{i=0}^{\infty} v_i(x,y,z,t)$ given in Equation (24) we get

$$\beta_0 = \frac{v_1}{v_0} = 7.79581 \times 10^{-8} < 1,$$
$$\beta_1 = \frac{v_2}{v_1} = 0.7341 < 1,$$
$$\beta_2 = \frac{v_3}{v_2} = 0.237846 < 1,$$
$$\beta_3 = \frac{v_4}{v_3} = 0.0718699 < 1,$$

where, the β_i values for $i \geq 0$ and $\forall (x,y,z,\ t):\ x,y,z \in \mathbb{R}^3,\ 0 < x,y,z,t \leq 1$ are less than 1, so the proposed iterative methods satisfy the convergence.

To examine the accuracy of the approximate solutions for this example, we to calculate the absolute error of the approximate solution, where the exact solution is $u = txyz$. The results are presented in Figures 5 and 6. The Figures show the absolute error $Absr_n$ for the approximate solution obtained by the proposed iterative methods and BDIM. We note that by increasing the number of iterations, the error decreases and the accuracy of the approximate solutions is increased. Shortening of the time step has a similar effect for BDIM.

In order to study the accuracy of the proposed methods, we measure the difference between the exact and numerical solution in terms of the RMS norm. The RMS norm is defined as $\mathrm{RMS} = \sqrt{\frac{\sum (e_i - n_i)^2}{\sum (e_i)^2}}$.

Here e_i is the exact solution in node i and n_i is the numerical solution at the same node at a certain time. This allows us to display RMS time diagrams for Examples 2, 4 and 6 in Figures 7–9. Iterative methods BCM, DJM and TAM have similar RMS difference properties. The accuracy is very high at the beginning of the simulation. For long periods of time the accuracy deteriorates. Since these approaches lead to an expansion of the solution if we increase the time, we actually go further away from the initial point. Therefore, the accuracy decreases as with the Taylor expansion. At the beginning of the simulation, when the accuracy is better than 10^{-15}, we notice some oscillations in the accuracy of the TAM method. The accuracy of the BDIM method is not dependent on time, but is defined by the mesh size and the length of the time steps. The best results are obtained with a short time step and a dense mesh. Because of these properties, the BDIM is more accurate than iterative methods for long periods of time.

It is worth mentioning that the main advantage of using TAM, DJM and BCM compared to other numerical methods is that no linearization or discretization is required, thus avoiding the large computational effort and rounding errors. The implementation does not include a restrictive

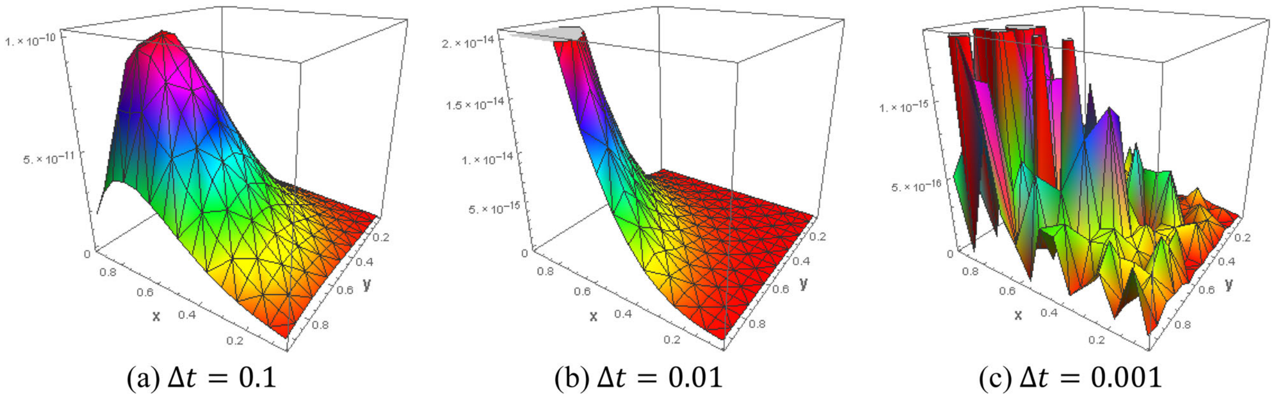


Figure 5. Absolute errors of the solution of Example 4 obtained by the BDIM using 17^3 equidistant nodes and three different time steps (0.1, 0.01 and 0.001). Results are shown on the $z = 0.1$ plane at $t = 0.1$. We observe that the error decreases with shortening of the time step.

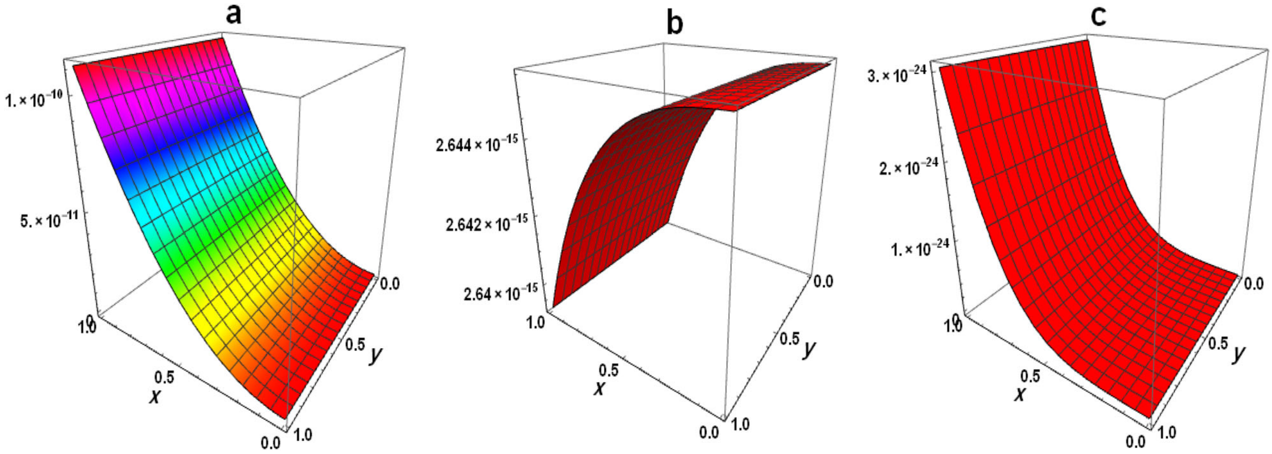


Figure 6. Absolute error $Absr_n$ of the solution of Example 4 obtained by the iterative methods for different number of iterations $n = 1, 3, 4$. The panels show $u_1(x,y,z,t)(a)$, $u_3(x,y,z,t)(b)$ and $u_4(x,y,z,t)$ (c) at time $t = 0.1$ and $z = 0.1$. We observe the increase of accuracy when the number of iterations is increased.

assumption for the nonlinear terms and it overcomes the difficulties encountered in the calculation of Adomian polynomials to handle the nonlinear terms, which is a disadvantage of the Adomian Decomposition Method (ADM). It does not require calculation of the Lagrange multiplier in Variational Iteration Method (VIM), where the terms of the sequence become complex after several iterations, so that the analytical evaluation of the terms becomes very difficult or impossible. There is also no need to construct a homotopy in Homotopy Perturbation Method and solve the corresponding algebraic equations.

5. Summary and conclusion

In this work, we developed three iterative methods TAM, DJM and BCM and a discretization-based BDIM method to find approximate solutions for the wave equation in 1D, 2D and 3D. The iterative methods provide the solutions in the form of a series. The accuracy of the solutions has been investigated by absolute error diagrams and the study of RMS error propagation in time. The convergence of the methods was investigated and the efficiency and accuracy was demonstrated.

We have shown that the accuracy of TAM, DJM and BCM increases with the number of iterations used and decreases over time when a constant number of iterations is used. From this we conclude that the number of iterations chosen must correspond to the time when the solution is needed. We have compared the accuracy of the iterative methods with BDIM, which is a domain-based method. We could achieve better accuracy with iterative methods as long as the number of iterations was large enough. On the other hand, we have observed that the accuracy of BDIM depends strongly on the grid discretization and the time step. The choice of a fine grid and a short time step leads to a better accuracy, but results in an increased computational effort.

Disclosure statement

No potential conflict of interest was reported by the authors.

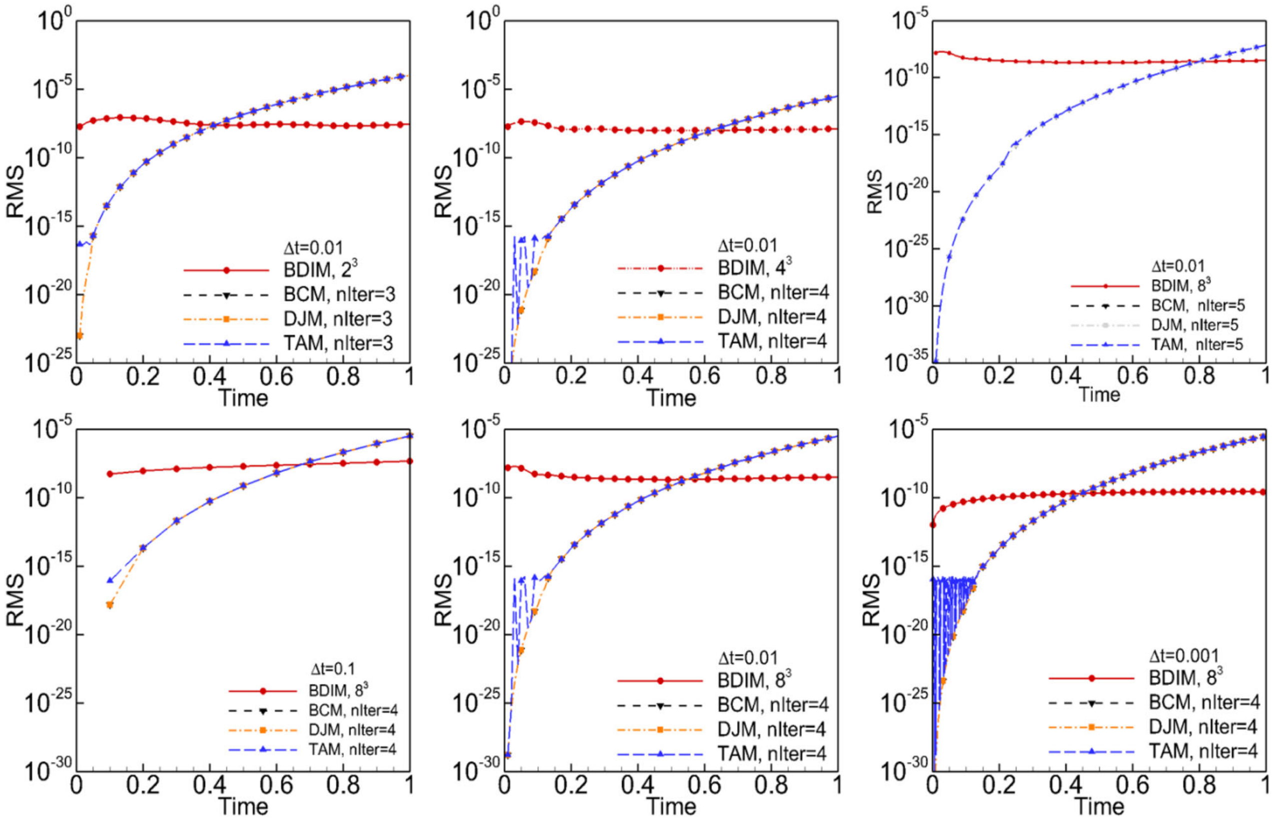


Figure 7. Plots of the RMS difference versus time for the solution of Example 2. Top row: dependence of mesh density (BDIM) and the number of iterations (BCM, DJM and TAM). Bottom row: dependence of time step size.

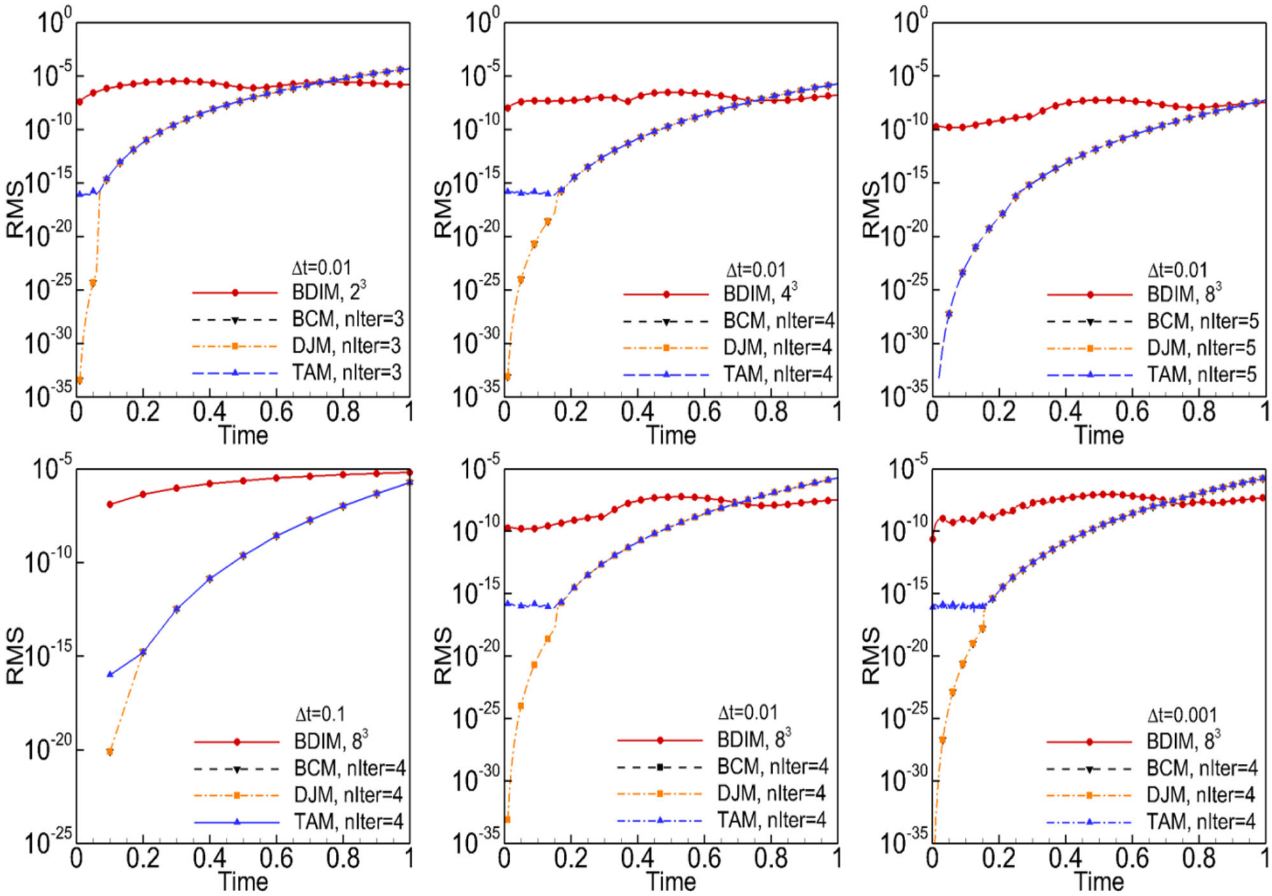


Figure 8. Plots of the RMS difference versus time for the solution of Example 4. Top row: dependence of mesh density (BDIM) and the number of iterations (BCM, DJM and TAM). Bottom row: dependence of time step size.

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