Arab Journal of Basic and Applied Sciences

ISSN: (Print) 2576-5299 (Online) Journal homepage: http://www.tandfonline.com/loi/tabs20

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To cite this article: Majeed Ahmed Al-Jawary, Ghassan Hasan Radhi & Jure Ravnik (2018) Development of the Banach contraction method for the solution of nonlinear thin film flows of non-Newtonian fluids, Arab Journal of Basic and Applied Sciences, 25:3, 122-131, DOI: 10.1080/25765299.2018.1511079

To link to this article: https://doi.org/10.1080/25765299.2018.1511079

Published online: 01 Dec 2018.

ARAB JOURNAL OF BASIC AND APPLIED SCIENCES
2018, VOL. 25, NO. 3, 122–131
https://doi.org/10.1080/25765299.2018.1511079

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Development of the Banach contraction method for the solution of nonlinear thin film flows of non-Newtonian fluids

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ABSTRACT

The aim of this study is to present a new implementation of an efficient iterative method proposed by Daftardar-Gejji and Bhalekar to solve a nonlinear initial value problem. We consider the thin film problem of a non-Newtonian fluid on a moving belt. The proposed method is based on using the Banach's contraction principle. In order to determine the accuracy of the obtained approximate solutions, several comparisons were done against solutions obtained by other authors using the Runge–Kutta method as well as the Newton–Raphson–Euler-based solution. Comparisons with the variational iteration method and the homotopy perturbation method were also made. Several measures have been adopted to provide the error analysis for the derived approximate solution: the error remainder with the maximal error remainder, the second norm and the root mean squared norm. The application of the proposed technique shows the purpose of the obtained approximate solution where no additional assumptions are needed for the nonlinear terms. Calculations were performed using the computer algebra system MATHEMATICA®10.

ARTICLE HISTORY
Received 31 March 2018
Revised 6 August 2018
Accepted 8 August 2018

KEYWORDS
Iterative method; Banach contraction principle; thin film flow; non-Newtonian fluid; analytic solution

1. Introduction

Viscosity is a property of a fluid, which governs diffusive transport of momentum during fluid flows. Newton's constitutive relation for fluids defines viscosity as the ratio between stress and strain rate in the flow. Fluids for which the Newton's constitutive relation holds are known as Newtonian fluids (Hu & Kieweg, 2012) in the engineering literature, and are characterized by a linear response to strain rate. In contrast to these, fluids which exhibit a non-linear response to strain rate are defined as non-Newtonian. Some examples of this kind of fluid are quicksand, asphalt, glue, cosmetics and others.

In recent years, research on the flows of non-Newtonian fluids has increased because of their wide applications in engineering. Many specialists in this field mention a wide range of applications that address the problems of rheology in various biological sciences, chemical industries, geophysics and others (Ellahi, Zeeshan, & Hassan, 2016; Ellahi, Tariq, Hassan, & Vafai, 2017; Ellahi, Zeeshan, Shehzad, & Alamri, 2018; Khan, Masood, Ellahi & Bhatti, 2018; Zhaosheng & Jianzhong, 1998; Zeeshan, Shehzad, & Ellahi, 2018).

On the other hand, we find that many researchers have found acceptable analytic solutions when studying the behaviour of non-Newtonian fluids (Ayub, Rasheed, & Hayat, 2003; Bougoffa, Duan, & Rach, 2015). One of the most prominent of these studies is the non-Newtonian Sisko fluid study. This fluid is modelled by a set of nonlinear equations, which allows scientists and researchers to examine different sets of parameters in the governing equations and find approximate solutions that are as close as possible to the exact solutions; for example the study of thin film flows of Sisko fluid on a moving belt (Sisko, 1958), considering the Sisko fluid flow in a collector, etc.

Some examples of methods that are used to study and solve this kind of nonlinear problem are the Adomian decomposition method (ADM) (Siddiqui, Farooq, Haroon, & Babcock, 2012), the homotopy analysis method (HAM) (Sajid & Hayat, 2008), the homotopy perturbation method (HPM) (Sajid, Hayat, & Asghar, 2007; Siddiqui, Mahmood, & Ghori, 2006), the variational iteration method (VIM) (Moosavi, Momeni, Tavangar, Mohammadyari, & Rahimi-Esbo, 2016; Siddiqui et al., 2012), and the Temimi and Ansari method (TAM) (AL-Jawary, 2017) and others.

In 2009, (Daftardar-Gejji & Bhalekar, 2009) Varsha Daftardar-Gejji and Sachin Bhalekar proposed an iterative technique which can be applied to different kinds of the nonlinear functional equations of the form $v = f + N(v)$. This iterative method is based on using the Banach contraction principle and it can be abbreviated as (BCPM). Daftardar and Bhalekar

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considered some examples to show the validity of the Banach contraction principle method (BCPM) in solving different kinds of equations (Daftardar-Gejji & Bhalekar, 2009). In this study, we will apply the BCPM to solve the nonlinear thin film flow problem, with a discussion about the error analysis and determining the accuracy of the obtained solutions and showing the high efficiency of the proposed method.

The following sections in this paper are organized as follows: section 2 presents the nonlinear thin film flow problem, section 3 presents several preliminaries for the BCPM, section 4 shows the basic steps of the BCPM, and in section 5 there is the convergence proof for the proposed method. Solving the nonlinear problem by the BCPM and the convergence analysis are given in section 6. The error analysis of the approximate solution is in section 7. Finally, a comparison study is presented in section 8.

2. The nonlinear thin film flow problem

In the following, we consider the nonlinear boundary problem, which represents the physical problem of the thin film flow of the third-grade fluid and on a moving belt (Siddiqui et al., 2006, 2012). First, the motion for the incompressible fluid can be governed by these basic equations:

$$\nabla . U = 0, \tag{1}$$

$$\rho \frac{DU}{Dt} = -\nabla p + \mathrm{div}\tau, \tag{2}$$

where U is the velocity vector, ρ is the constant density, p represents the pressure, D/Dt denotes the material derivative and τ is the stress tensor. These equations are valid for an incompressible fluid, neglecting all body forces and thermal effects. The stress tensor defines the following third-grade fluid presented by:

$$\tau = \sum_{k=1}^{3} Z_k, \tag{3}$$

where

$$\begin{gathered} Z_1 = \mu C_1, \\ Z_2 = \gamma_1 C_2 + \gamma_2 C_1^2, \\ Z_3 = a_1 C_3 + a_2(C_1 C_2 + C_2 C_1) + a_3(trC_2)C_1. \end{gathered} \tag{4}$$

Above, μ represents the viscosity constant and γ_1, γ_2, a_1, a_2 and a_3 are material constants of the problem. The Rivlin–Ericksen tensors C_n are defined by:

$$\begin{gathered} C_0 = \mathrm{I}, \\ C_n = \frac{DC_{n-1}}{Dt} + C_{n-1}(\nabla U) + (\nabla U)^T C_{n-1}, \quad n \geq 1. \end{gathered} \tag{5}$$

where C_0 is the identity tensor. Now let suppose a third-grade fluid (as in Figure 1) in a container where a wide moving belt passes it. This belt is moving vertically upward with the velocity constant a.

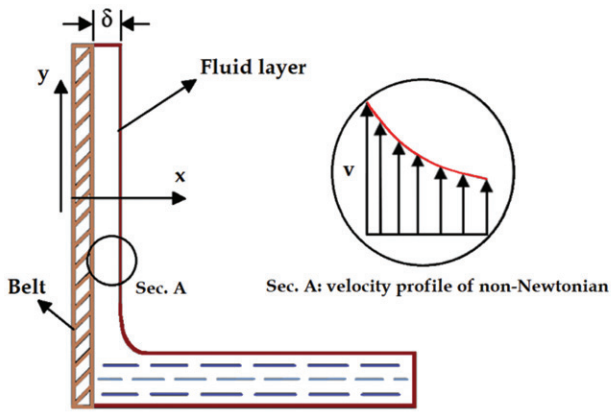

Figure 1. Physical sketch for the flow of a moving belt through a non-Newtonian fluid (Adomian, 1994; Moosavi et al., 2016).

As the belt is moving up and passing through the fluid, it is picking up a film of thickness δ. As a result of gravity the film drains down the belt. We consider the flow to be steady and uniform. We set up the coordinate system such that the x-axis is parallel to the fluid and is normal to the belt, the y-axis is aimed upward along the belt and the z-axis is normal to the xy-plane. This set up, where only the component of the velocity in the y-direction is non-zero, can be considered to take the following form:

$$U = (0, v(x), 0). \tag{6}$$

Such flow velocity (6) identically satisfies the continuity equation (1). Making use of equation (3) and (4), we can derive the components of the momentum equation (2) as:

$$\frac{\partial p}{\partial x} = (2\gamma_1 + \gamma_2)\frac{d}{dx}\left(\frac{dv}{dx}\right)^2, \tag{7}$$

$$\frac{\partial p}{\partial y} = \rho g + \mu \frac{d}{dx}\left(\frac{dv}{dx}\right) + 2(a_2 + a_3)\frac{d}{dx}\left(\frac{dv}{dx}\right)^3, \tag{8}$$

$$\frac{\partial p}{\partial z} = 0, \tag{9}$$

where ρ is the density and g is the acceleration due to gravity. Equation (9) reveals that the pressure p does not depend on the z coordinate, so it is a function of x and y only. By introducing the generalized pressure $\tilde{p}$ by the relation:

$$\tilde{p} = p - (2\gamma_1 + \gamma_2)\left(\frac{dv}{dx}\right)^2, \tag{10}$$

the equations (7) and (8) take these forms:

$$\frac{\partial \tilde{p}}{\partial x} = 0, \tag{11}$$

$$\frac{\partial \tilde{p}}{\partial y} = \rho g + \mu \frac{d^2 v}{dx^2} + 6(a_2 + a_3)\frac{d^2 v}{dx^2}\left(\frac{dv}{dx}\right)^2. \tag{12}$$

Equation (11) reveals that the pressure is independent of x, thus equation (12) becomes:

$$\frac{d\tilde{p}}{dy}=\rho g+\mu\frac{d^2v}{dx^2}+6(a_2+a_3)\frac{d^2v}{dx^2}\left(\frac{dv}{dx}\right)^2. \tag{13}$$

Each term on the right-hand side of equation (13) involves velocity, which depends on x, and on the left-hand side the pressure depends on y. Equality is possible only if both sides are constant. When there is no pressure in the y-direction, this constant takes the value zero and we have:

$$\frac{d^2v}{dx^2}+\frac{6(a_2+a_3)}{\mu}\left(\frac{dv}{dx}\right)^2\frac{d^2v}{dx^2}-\frac{\rho g}{\mu}=0, \tag{14}$$

$$v(0)=a,\ \ \frac{dv}{dx}(\delta)=0. \tag{15}$$

Some assumptions are made to simplify the problem: the film flow thickness δ is uniform and the flow is steady-state, laminar and uniform (Moosavi et al., 2016). In what follows, the dimensionless variables are introduced in the following way:

$$\tilde{x}=\frac{x}{\delta},\ \ \tilde{v}=\frac{v}{a},\ \ \beta=\frac{(a_2+a_3)a^2}{\mu\delta^2}\ \text{and}\ m=\frac{\rho g\delta^2}{\mu a}. \tag{16}$$

Hence, equations (14) and (15) can be reduced to the following well-posed boundary value problem, where the tilde symbol is omitted:

$$v''+6\beta(v')^2v''-m=0, \tag{17}$$

with the following boundary conditions:

$$v(0)=1,\ \ v'(1)=0. \tag{18}$$

By integrating both sides of (17) with respect to x, we may write:

$$v'+2\beta(v')^3-mx=c_1, \tag{19}$$

and by implementing the second boundary condition in equation (18), we can get $c_1=-m$. Thus, the next first-order ODE can be derived:

$$v'+2\beta(v')^3-m(x-1)=0, \tag{20}$$

$$v(0)=1. \tag{21}$$

In the case of considering the flow on a Newtonian fluid, the parameter β equals $\beta=0$ (Siddiqui et al., 2006, 2012). In the following sections, we will apply the BCPM for the nonlinear initial value problem, which is presented by equations (20) and (21) instead of the boundary value problem that is presented in equations (17) and (18), as we expect that the solution of the initial value problem will be more accurate and less computationally expensive.

3. Some preliminaries

In this section, we review some basic concepts.

Let X_1 and X_2 be two metric spaces, and let F be a mapping from X_1 into X_2, F is said to be Lipschitz mapping, if there exists a real number $r\geq 0$ such that for all $x_1,\ x_2\in X_1$ we have $d(Fx_1,Fx_2)\leq rd(x_1,x_2)$. F is said to be contraction mapping if $r<1$ (Joshi & Bose, 1985).

(Joshi & Bose, 1985) Let F be a contraction mapping with a Lipschitz constant r, of a complete metric space X into itself, then F has a unique fixed point v in the space X. In addition, if x_0 is some arbitrary point in X, and x_n is defined by $x_{n+1}=F(x_n)$, $n=0,1,2,\ldots$ then $\lim_{n\to\infty}x_n=v$ and $d(x_n,v)\leq\frac{r^n}{1-r}d(x_1,x_0)$.

(Joshi & Bose, 1985) Let F be a mapping of some complete metric space X into itself, such that F^k is a contraction mapping of X for a positive integer k, then F has a unique fixed point in the space X.

4. The analysis of the BCPM

Let us consider the following nonlinear equation:

$$v(x)=f(x)+N[v(x)], \tag{22}$$

where $v(x)$ is an unknown function, $f(x)$ is a given function and N is a nonlinear operator of the functional equation (22). Let us define some successive approximations as the following:

$$\begin{aligned} v_0&=f,\\ v_1&=v_0+N[v_0],\\ v_2&=v_0+N[v_1],\\ &\vdots \end{aligned} \tag{23}$$

$$v_n=v_0+N[v_{n-1}],\ \ n=1,2,\ldots. \tag{24}$$

If N^k is a contraction mapping for some positive integer k, so $N[v]$ has a unique fixed point and thus the nth sequence that is defined by (24) is convergent according to Theorem 3.2, hence the solution of (22) will be obtained as the following form:

$$v=\lim_{n\to\infty}v_n. \tag{25}$$

5. The convergence proof for the BCPM

In order to show the convergence proof for the BCPM, let us start by defining the algorithm for the proposed method. The iterative algorithm starts with these terms:

$$\begin{aligned} V_0&=v_0(x),\\ V_1&=F[v_0],\\ V_2&=F[v_0+v_1],\\ &\ldots\\ V_{n+1}&=F[v_0+v_1+\ldots+v_n], \end{aligned} \tag{26}$$

where F is an operator which can be defined by:

$$F(V_k) = S_k - \sum_{i=0}^{k-1} V_i(x), \ k = 1, 2, \ldots. \tag{27}$$

Here S_k represents the problem solution, which is obtained by the BCPM in the form:

$$V_k = V_0 + N\left(\sum_{i=0}^{k-1} V_i(x)\right), \ k = 1, 2, \ldots. \tag{28}$$

This iterative process yields $v(x) = \lim_{i\to\infty} v_i(x) = \sum_{i=0}^{\infty} V_i(x)$. Thus, when using equations (26) and (27), the solution will result in a series of the following form:

$$v(x) = \sum_{i=0}^{\infty} V_i(x). \tag{29}$$

According to the proposed BCPM iterative algorithm, the sufficient convergence condition will hold for this technique, where the main results are presented in the following theorems.

Let F, given in (27), be an operator from a Hilbert space H to H. The series solution $v_n(x) = \sum_{i=0}^{n} V_i(x)$ converges if $\exists\ 0 < \eta < 1$ such that $\|F[V_0 + V_1 + \ldots + V_{i+1}]\| \le \eta \|F[V_0 + V_1 + \ldots + V_i]\|$ (that is $\|V_{i+1}\| \le \eta\|V_i\|$) $\forall i = 0, 1, 2, \ldots$.

This theorem is a special case of Banach's fixed point theorem and it is a sufficient condition for studying the convergence of the BCPM.

See (Odibat, 2010).

If the series solution $v(x) = \sum_{i=0}^{\infty} V_i(x)$ is convergent, then this series will represent the exact solution for the nonlinear problem defined by equations (20) and (21).

See (Odibat, 2010).

Suppose that the series solution $\sum_{i=0}^{\infty} V_i(x)$ which is defined in (29) is convergent to the solution $v(x)$. If the truncated series $\sum_{i=0}^{n} V_i(x)$ is used as an approximation to the solution of problem defined by (20) and (21), then the maximum error $E_n(x)$ will be estimated as:

$$E_n(x) \le \frac{1}{1-\eta}\eta^{n+1}\|V_0\|. \tag{30}$$

See (Odibat, 2010).

To summarize, Theorems 5.1 and 5.2 state that the BCPM solution for the nonlinear equation (20), evaluated by (24) or (26), converges to the exact solution under the condition $\exists\ 0 < \eta < 1$ such that $\|F[V_0 + V_1 + \ldots + V_{i+1}]\| \le \eta\|F[V_0 + V_1 + \ldots + V_i]\|$ (that is $\|V_{i+1}\| \le \eta\|V_i\|$) $\forall i = 0, 1, 2, \ldots$. In other words, for each i, if we provide the following parameters:

$$\alpha_i = \begin{cases} \dfrac{\|V_{i+1}\|}{\|V_i\|}, & \|V_i\| \ne 0 \\ 0, & \|V_i\| = 0 \end{cases} \tag{31}$$

then the series solution $\sum_{i=0}^{\infty} V_i(x)$ for equation (20) converges to the exact solution $v(x)$, when $0 \le \alpha_i < 1,\ \forall i = 0, 1, 2, \ldots$. Also, as in Theorem 5.3, the maximum truncation error is estimated to be $\|v(x) - \sum_{i=0}^{n} V_i\| \le \frac{1}{1-\alpha}\alpha^{n+1}\|V_0\|$, where $\alpha = \max\{\alpha_i, i = 0, 1, \ldots, n\}$.

6. Solving the nonlinear thin film flow problem by using the BCPM with convergence

In this section, we develop the BCPM iterative method to solve the nonlinear thin film flow problem. In the following, we consider the nonlinear initial value problem that is represented by equations (20) and (21).

According to the BCPM, we start by integrating both sides of equation (20) from 0 to x. The integration yields the following expression:

$$v(x) = 1 - mx + m\frac{x^2}{2} - 2\beta\int_0^x \left(v'(t)\right)^3 dt. \tag{32}$$

By selecting the initial approximation v_0, which is:

$$v_0(x) = 1 - mx + m\frac{x^2}{2}, \tag{33}$$

the first approximation v_1 can be written in the following form:

$$v_1(x) = v_0(x) - 2\beta\int_0^x \left(v_0'(t)\right)^3 dt, \tag{34}$$

$$\begin{aligned} v_1(x) =& 1 - mx + \frac{mx^2}{2} + 2m^3x\beta - 3m^3x^2\beta \\ &+ 2m^3x^3\beta - \frac{1}{2}m^3x^4\beta. \end{aligned} \tag{35}$$

Similarly, the second approximation v_2 reads as:

$$v_2(x) = v_0(x) - 2\beta\int_0^x \left(v_1'(t)\right)^3 dt, \tag{36}$$

$$\begin{aligned} v_2(x) = 1 - mx + \frac{mx^2}{2} &+ 2m^3x\beta - 3m^3x^2\beta \\ &+ 2m^3x^3\beta - \frac{1}{2}m^3x^4\beta - 12m^5x\beta^2 + 30m^5x^2\beta^2 \\ &- 40m^5x^3\beta^2 + 30m^5x^4\beta^2 - 12m^5x^5\beta^2 + 2m^5x^6\beta^2 \\ &+ 24m^7x\beta^3 - 84m^7x^2\beta^3 + 168m^7x^3\beta^3 - 210m^7x^4\beta^3 \\ &+ 168m^7x^5\beta^3 - 84m^7x^6\beta^3 + 24m^7x^7\beta^3 - 3m^7x^8\beta^3 \\ &- 16m^9x\beta^4 + 72m^9x^2\beta^4 - 192m^9x^3\beta^4 + 336m^9x^4\beta^4 \\ &- \frac{2016}{5}m^9x^5\beta^4 + 336m^9x^6\beta^4 - 192m^9x^7\beta^4 \\ &+ 72m^9x^8\beta^4 - 16m^9x^9\beta^4 + \frac{8}{5}m^9x^{10}\beta^4. \end{aligned} \tag{37}$$

We can obtain the rest of the approximations using the same process. In general, the following formula is used for this purpose:

$$v_{n+1}(x) = v_0(x) - 2\beta\int_0^x \left(v_n'(t)\right)^3 dt, \ n = 0, 1, 2 \ldots. \tag{38}$$

In order to check the convergence of the BCPM for the current initial value problem presented by (20) and (21), as given in the relations (26)–(29), the iterative scheme for equation (20) can be formulated by:

$$V_0(x) = v_0(x) = 1 - mx + m\frac{x^2}{2}, \tag{39}$$

with the operator $F[V_k]$ as defined in equation (27) with the term S_k which represents the solution for the following problem:

$$V_k = V_0 - 2\beta\int_0^x \left(\frac{d}{dx}\left(\sum_{i=0}^{k-1} V_i(x)\right)\right)^3 dt,\quad k = 1, 2, \ldots. \tag{40}$$

The following terms are obtained:

$$V_1 = -\frac{m^3}{2}\beta\left(1-(x-1)^4\right), \tag{41}$$

$$\begin{aligned}V_2 =& \frac{1}{5}m^5(-2+x)x\beta^2(-64m^4x^7\beta^2 + 8m^4x^8\beta^2\\ &+ 2m^2x^5\beta(45 - 248m^2\beta) + m^2x^6\beta(-15 + 232m^2\beta)\\ &+ 10(3 - 6m^2\beta + 4m^4\beta^2) - 20x\left(3-9m^2\beta + 8m^4\beta^2\right)\\ &- 40x^3\left(1-9m^2\beta + 16m^4\beta^2\right) + 10x^2\left(7-33m^2\beta + 40m^4\beta^2\right)\\ &+ 2x^4\left(5-120m^2\beta + 344m^4\beta^2\right)).\end{aligned} \tag{42}$$

As given in the convergence proof for the BCPM, the terms given by the series $\sum_{i=0}^{\infty} V_i(x)$ in (29) satisfy the convergence condition. By evaluating the α_i values, we get:

$$\begin{aligned}\alpha_0 &= \frac{\|V_1\|}{\|V_0\|} = 0.00658939 < 1\\ \alpha_1 &= \frac{\|V_2\|}{\|V_1\|} = 0.176445 < 1\\ \alpha_2 &= \frac{\|V_3\|}{\|V_2\|} = 0.180917 < 1\\ \alpha_3 &= \frac{\|V_4\|}{\|V_3\|} = 0.192138 < 1\\ \alpha_4 &= \frac{\|V_5\|}{\|V_4\|} = 0.197802 < 1\end{aligned} \tag{43}$$

where the α_i values for $i \geq 0$ and $0 < x \leq 1$ are less than 1, therefore the BCPM satisfies the convergence condition, and this makes the BCPM a valid mathematical tool for solving such kind of problems. In order to estimate the accuracy of the final approximate solution, we define the following error remainder function:

$$ER_n(x) = \frac{d}{dx}(v_n) + 2\beta\left(\frac{d}{dx}(v_n)\right)^3 - m(x-1) = 0, \tag{44}$$

and the following maximal error remainder parameter:

$$MER_n = \max_{0\leq x\leq 1}\left|ER_n(x)\right|. \tag{45}$$

All the computations presented in this paper have been conducted using the symbolic computation software MATHEMATICA®.

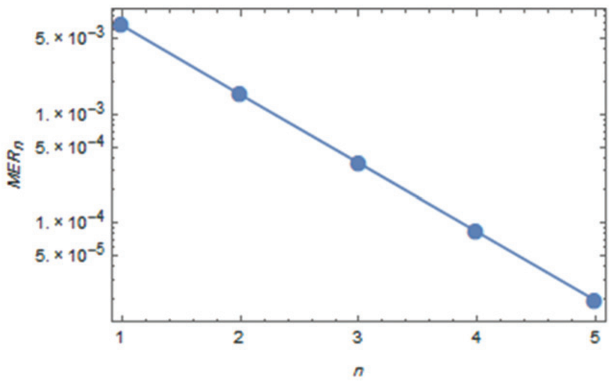

Figure 2. The maximum error remainder plots for the thin film flow problem by using the BCPM where $\beta = 0.5$ and $m = 0.3$.

All terms in the BCPM approximation series involve β and its powers, which makes a noticeable impact on the behaviour of the non-Newtonian fluid problem. By setting the value of β as zero the Newtonian viscous fluid problem will be recovered and the BCPM series will converge to the exact solution. In the next section a numerical simulation for different cases of the flow problem will be discussed.

7. The numerical simulation

When studying the numerical solution of the problem posed by equations (20) and (21), one must examine the effect of the BCPM on the convergence of the function $v(x)$. From this standpoint, we must examine the numerical structure of the obtained approximate solution $v(x)$. Hence, when choosing the values of β and m in our approximate solution, we can obtain several approximate solutions. The best solution is obtained when setting $\beta = 0.5$ and $m = 0.3$ just like in AL-Jawary (2017) and Siddiqui et al. (2012). The BCPM approximations for this case are:

$$v_0(x) = 1 + 0.3\left(-1 + \frac{x}{2}\right)x,$$

$$\begin{aligned}v_1(x) =& 1. - 0.27299999999999996x\\ &+ 0.10949999999999999x^2 + 0.027x^3 - 0.00675x^4,\end{aligned}$$

$$\begin{aligned}v_2(x) =& 1. - 0.279653583x + 0.1255172235x^2\\ &+ 0.007056503999999997x^3 + 0.006147468000000006x^4\\ &- 0.003193311599999992x^5 - 0.0006680070000000001x^6\\ &+ 0.0004199039999999999x^7 + 0.00000656100000000002x^8\\ &- 0.000019682999999999998x^9 + 0.0000019683x^{10}.\end{aligned}$$

It can be observed that when drawing the values of the MER_n as in Figure 2, the obtained solution by the BCPM has an exponential rate of convergence, since the points are located on a straight line.

In order to additionally verify the accuracy of BCPM solutions of the flow problem, we have solved the problem by using the classical fourth-order Runge–Kutta method (RKM), which is implemented

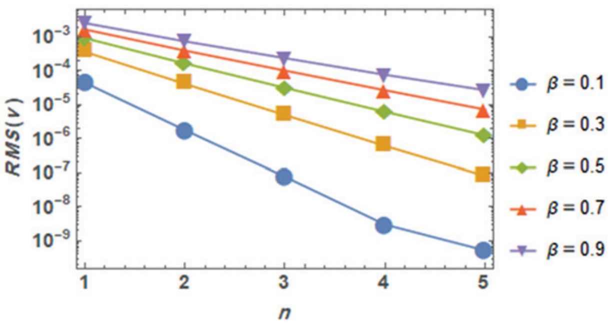

Figure 3. The root mean square $RMS(v)$ curves that are obtained by the BCPM, for different values of β and $m = 0.3$.

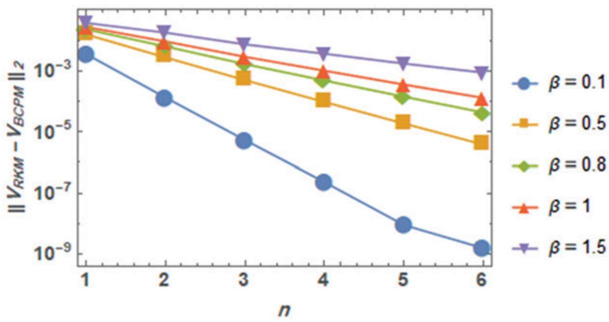

Figure 5. Comparison of the solutions of the BCPM and the RKM by using the 2-norm $\|v_{\mathrm{RKM}} - v_{\mathrm{BCPM}}\|_2$ at $m = 0.3$.

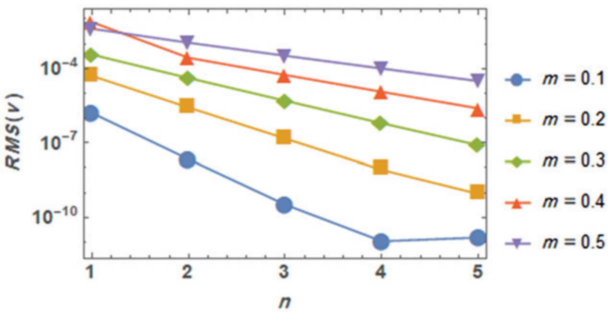

Figure 4. The root mean square $RMS(v)$ curves that are obtained by the BCPM, for different values of m and $\beta = 0.3$.

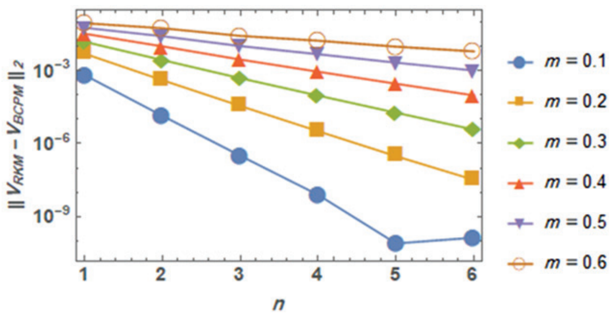

Figure 6. Comparison of the solutions of the BCPM and the RKM by using the 2-norm $\|v_{\mathrm{RKM}} - v_{\mathrm{BCPM}}\|_2$ at $\beta = 0.5$.

in MATHEMATICA (see appendix in (AL-Jawary, 2017)). The RKM can be used as a benchmark to assess the performance of the BCPM. A similar approach was used by AL-Jawary (2017) and Siddiqui et al. (2012). To show the ability of the BCPM for reaching the best accuracy for the obtained approximate solutions, we used the root mean square (RMS) function to estimate the difference between the solutions of the BCPM and RKM. The RMS function is defined in the following way:

$$RMS(v) = \sqrt{\frac{\sum (v_{\mathrm{BCPM}} - v_{\mathrm{RKM}})^2}{\sum (v_{\mathrm{RKM}})^2}}. \tag{46}$$

Figures 3 and 4 show the $RMS(v)$ differences versus n, where the RMS value decreases whenever the value of n increases. Note that the higher the non-Newtonian parameter β, the RMS values becomes greater; where the value of the constant m is fixed as shown in Figure 3. The same conclusion can also be drawn when keeping the non-Newtonian parameter β fixed and increasing the value of m as shown in Figure 4. We conclude from this that the approximate solutions of the BCPM become more accurate whenever n increases. The rate of convergence with increasing n for the case of $\beta = 0.5$ and $m = 0.3$ was estimated using $\log(MER_4/MER_3)/\log(MER_3/MER_2) = 1.0$ proving linear convergence of the method.

In order to highlight the difference between solutions; we have evaluated the 2-norm, i.e. $\|v_{\mathrm{RKM}} - v_{\mathrm{BCPM}}\|_2$ for five approximations of the BCPM. We observe good accuracy when changing the non-Newtonian parameter β and keeping the value of m fixed, as shown in Figure 5. The same conclusion can be drawn when retaining the value of β fixed, and changing the value of the constant m, as in Figure 6.

The BCPM is comprehensive and smooth in solving various types of problems. In comparison with the RKM, we can find that in using the BCPM there is no need for any restrictive assumptions, resorting to discretization or determining the step size of the subintervals over the whole interval. In addition, the use of the BCPM does not require a large and complicated computational work or using any type of quantization processes.

In order to additionally verify and validate the proposed BCPM approach, we solved the problem using a combination of Euler and Newton–Raphson methods (ENRM). We have discretizised the solution space into steps of size Δx. At each step, we solved the nonlinear equation (20) for the given x value and obtained v' using the Newton–Raphson method. Then we advanced the solution v in space using the Euler method. The Newton–Raphson method is implemented as:

$$f(v') = v' + 2\beta(v')^3 - m(x-1) \text{ and } f'(v') = 1 + 6\beta(v')^2 \tag{47}$$

$$v' = v'^{(k)} + \frac{f(v'^{(k)})}{f'(v'^{(k)})}. \tag{48}$$

We iterate equation (48) until convergence is achieved $(v'^{(k+1)} - v'^{(k)} < 10^{-13})$. With v' calculated at a

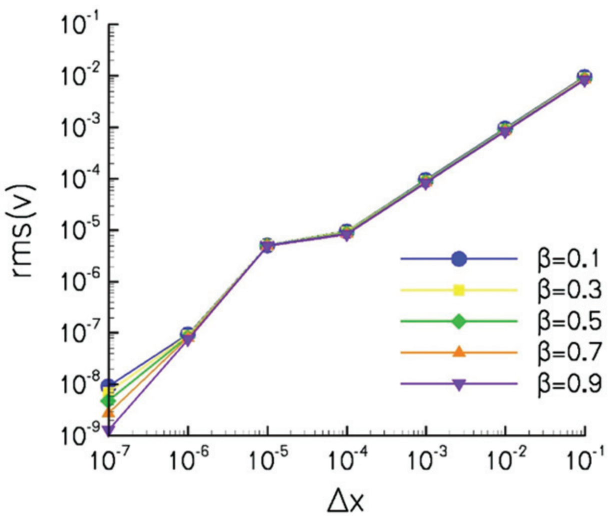

Figure 7. The root mean square norm $RMS(v)$ versus the spatial discretization for different values β and $m = 0.3$.

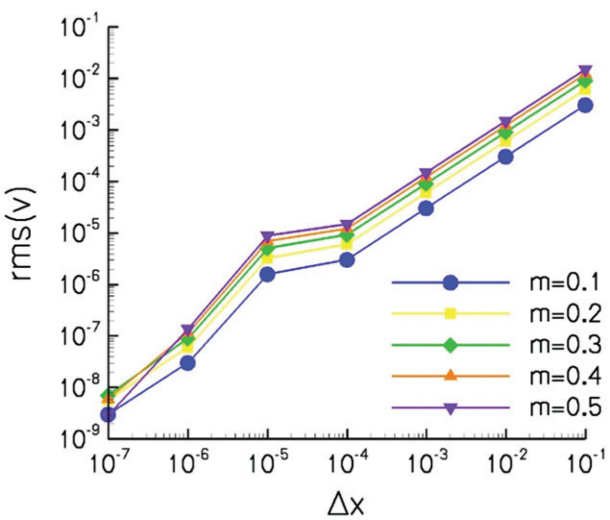

Figure 8. The root mean square norm $RMS(v)$ versus the spatial discretization for different values m and $\beta = 0.3$.

chosen x, we advance the solution using $v(x + \Delta x) = v(x) + v'\Delta x$.

Figures 7 and 8 show the RMS norm of ENRM versus the RKM. We observe the solution accuracy is increasing when small discretization steps are used. In order to achieve solution accuracy comparable to the BCPM a very small step Δx must be used, and consequently a lot of calculations are needed. However, in the more demanding cases (for example for $m = 0.5$ or $\beta = 0.9$), the ENRM method yields results of the same order of accuracy for a given step size, while we observe decreased accuracy at a given n when using BCPM.

8. A comparison study

This section presents a comparison between the BCPM solution and other several previous solutions. These solutions have been evaluated by the ADM and VIM (Siddiqui et al., 2012) for the nonlinear thin film flow problem, i.e. equations (20) and (21). In the following we summarize ADM and VIM for the proposed nonlinear problem.

8.1. *Solution obtained by the Adomian decomposition method*

The basic details of the ADM can be found in (Adomian, 1994). Let us re-write equation (20) in the operator form:

$$L(v) + 2\beta N(v) + f(x) = 0, \tag{49}$$

where $L(v) = v'$ which is the highest order derivative, $N(v) = (v')^3$ is the nonlinear tem of equation (20) and $f(x) = -m(x-1)$, which is source term. The basic idea of the ADM is to expand the solution $v(x)$ in the following infinite series:

$$v(x) = \sum_{i=0}^{\infty} v_i(x), \quad i = 0, 1, 2, \ldots. \tag{50}$$

The components v_i are calculated in recursive way. The nonlinear term $N(v)$ is decomposed into the following infinite polynomial:

$$N(v) = \sum_{i=0}^{\infty} A_i, \tag{51}$$

where A_i are known by the Adomian polynomials that can be defined by (Adomian, 1994):

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N\left(\sum_{j=0}^{i} \lambda^j v_j\right)\right]_{\lambda=0}, \quad i = 0, 1, 2, \ldots. \tag{52}$$

The recurrence relation for the nonlinear initial value problem presented by (20) and (21) can be given by:

$$v_0(x) = 1 + mL^{-1}(x-1), \tag{53}$$

$$v_{i+1}(x) = 2\beta L^{-1}A_i, \quad i = 0, 1, 2, \ldots, \tag{54}$$

where the operator $L^{-1}(\cdot) = \int_0^x (\cdot)dt$. The first terms for the Adomian polynomials A_i are presented by:

$$\begin{aligned} A_0 &= v_0'^3, \\ A_1 &= 3v_0'^2 v_1', \\ A_2 &= 3v_0' v_1'^2 + 3v_0'^2 v_2', \\ A_3 &= v_1'^3 + 6v_0' v_1' v_2' + 3v_0'^2 v_3', \\ A_4 &= 3v_1'^2 v_2' + 3v_0' v_2'^2 + 6v_0' v_1' v_3' + 3v_0'^2 v_4'. \end{aligned} \tag{55}$$

Thus, the following components are obtained:

$$\begin{aligned} v_0(x) &= 1 - \frac{m}{2}\left(1-(x-1)^2\right), \\ v_1(x) &= -\frac{m^3}{2}\beta\left(1-(x-1)^4\right), \\ v_2(x) &= -2m^5\beta^2\left(1-(x-1)^6\right), \\ v_3(x) &= 12m^7\beta^3\left(1-(x-1)^8\right), \\ v_4(x) &= -88m^9\beta^4\left(1-(x-1)^{10}\right), \\ v_5(x) &= 728m^{11}\beta^5\left(1-(x-1)^{12}\right), \\ &\ldots \end{aligned} \tag{56}$$

The approximate solution $\psi_5 = \sum_{i=0}^{5} v_i(x)$ has been used as a solution obtained by the ADM.

This solution has been evaluated when setting $\beta = 0.5$ and $m = 0.3$ as in AL-Jawary (2017) and Siddiqui et al. (2012); so, we have:

$$\begin{aligned} v_{ADM}(x) = {} & 1. - 0.27826455369x + 0.12075128029499997x^2 \\ & + 0.016946227350000002x^3 - 0.008703601537499994x^4 \\ & + 0.01571850845999999x^5 - 0.02247460586999999x^6 \\ & + 0.021551966459999992x^7 - 0.015405474037499995x^8 \\ & + 0.007783642349999997x^9 - 0.002551605704999999x^{10} \\ & + 0.00048361130999999985x^{11} \\ & - 0.000040300942499999985x^{12}. \end{aligned} \tag{57}$$

8.2. *Solution obtained by the variational iteration method*

To examine the details of the VIM, please refer to He (1999). VIM considers equation (49) and uses the correction functional, which is defined by:

$$v_{i+1}(x) = v_i(x) + \int_0^x \lambda(Lv_i(t) + 2\beta N\tilde{v}_i(t) + f(t))dt, \quad i = 0, 1, 2, \ldots, \tag{58}$$

where λ represents the Lagrange multiplier, which can be selected by the variational theory (He, 1999). For our problem $\lambda = -1$ is chosen, since it is an ODE of the first order. The term $\tilde{v}_i(x)$ is the restricted variation, i.e. $\delta\tilde{v}_i = 0$. After determining the Lagrange multiplier λ, the correction functional will be:

$$v_{i+1}(x) = v_i(x) - \int_0^x \left(v_i'(t) + 2\beta\left(v_i'(t)\right)^3 - m(t-1)\right)dt, \quad i = 0, 1, 2, \ldots. \tag{59}$$

Now, selecting the initial function $v_0 = 1$ and by using the given initial condition in equation (21), the following approximate iterations are obtained:

$$v_1(x) = 1 - mx + \frac{mx^2}{2}, \tag{60}$$

$$v_2(x) = 1 - mx + \frac{mx^2}{2} - \frac{1}{2}m^3\left(-1 + (-1 + x)^4\right)\beta, \tag{61}$$

$$\begin{aligned} v_3(x) = {} & 1 + \frac{1}{2}m(-2 + x)x - \frac{1}{2}m^3x(-4 + 6x - 4x^2 + x^3)\beta \\ & + 2m^5x(-6 + 15x - 20x^2 + 15x^3 - 6x^4 + x^5)\beta^2 \\ & - 3m^7x(-8 + 28x - 56x^2 + 70x^3 - 56x^4 + 28x^5 - 8x^6 + x^7)\beta^3 \\ & + \frac{8}{5}m^9x(-10 + 45x - 120x^2 + 210x^3 - 252x^4 \\ & + 210x^5 - 120x^6 + 45x^7 - 10x^8 + x^9)\beta^4. \end{aligned} \tag{62}$$

The fifth iteration has been evaluated but for brevity it has not be written. When $\beta = 0.5$ and $m = 0.3$ are chosen, we obtain in the fifth iteration the following expression:

$$\begin{aligned} v_{VIM}(x) = {} & 1. - 0.27848503783412076x \\ & + 0.12202390679789032x^2 \\ & + 0.012462212476985899x^3 \\ & + 0.0020497509517549773x^4 \\ & - 0.0027938529654141404x^5 \\ & + 0.0010082798100599286x^6 \\ & - 0.0005743472921795505x^7 \\ & - 0.00005080525431622063x^8 \\ & + 0.00016705930916772292x^9 \\ & - 0.000028418954191598908x^{10} \\ & + 0.000008455254093027658x^{11} \\ & - 0.0000009550251293606744x^{12} + \ldots. \end{aligned} \tag{63}$$

Tables 1 and 2 review the maximum error remainder MER_5 for our solution of the BCPM and the previous solutions obtained by the ADM, VIM and the TAM. Clearly the best accuracy is reached by using the BCPM.

Table 1. The error norm MER_5 for the numerical solutions obtained by the BCPM, ADM, VIM and TAM for different values of β at $m = 0.3$.

β	BCPM	ADM	VIM	TAM
0.1	2.11964×10^{-9}	1.39694×10^{-8}	4.06296×10^{-8}	4.06296×10^{-8}
0.2	1.17068×10^{-7}	8.59592×10^{-7}	1.15823×10^{-6}	1.15823×10^{-6}
0.3	1.16024×10^{-6}	9.43745×10^{-6}	7.88187×10^{-6}	7.88187×10^{-6}
0.4	5.71314×10^{-6}	0.0000512	0.0000299	0.0000299
0.5	0.0000192	0.000189	0.0000826	0.0000826
1	0.000707	0.010431	0.001694	0.001694

Table 2. The error norm MER_5 for the numerical solutions obtained by the BCPM, ADM, VIM and TAM for different values of m at $\beta = 0.5$.

m	BCPM	ADM	VIM	TAM
0.1	2.22449×10^{-11}	1.39442×10^{-10}	7.56112×10^{-10}	7.56112×10^{-10}
0.2	1.42288×10^{-7}	1.06931×10^{-6}	1.27554×10^{-6}	1.27554×10^{-6}
0.3	0.0000192	0.000189099	0.0000826431	0.0000826431
0.4	0.000521409	0.00708853	0.00137525	0.00137525
0.5	0.00584935	0.108326	0.0107724	0.0107724

Conclusion

In this work, we have introduced a semi-analytical iterative method to solve the nonlinear thin film flow problem. This iterative method uses the Banach contraction principle theorem and there is no need to use any restricted assumptions when dealing with the nonlinear terms. When comparing this iterative method with the other known iterative methods such as the ADM, HAM, VIM and HPM, we find that this method is more easily implemented than those methods when solving nonlinear problems, as there is no need to use any additional or supportive calculations. The BCPM method is characterized by computational efficiency and there is no need to produce new more intricate approximations. In a numerical simulation we have shown that the values of the maximum error remainder decrease when the number of the BCPM iterations increases. Furthermore, the numerical results of the BCPM were compared with those obtained by the RKM by evaluating the root mean square norm. In addition, to confirm the accuracy of our numerical calculations of the BCPM, we have evaluated the 2-norm for the velocity function based on the values of the BCPM and RKM. The figures show linear convergence of the method. Thus, the BCPM is a very accurate method for finding reliable results with high accuracy. Comparison with the Euler–Newton–Raphson approach showed that in most cases BCMP is computationally faster and more efficient. Calculations were performed using the computer algebra system MATHEMATICA®.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

Adomian, G. (1994), *Solving Frontier Problems of Physics: The Decomposition Method*. Boston, MA: Kluwer Academic Publishers.

AL-Jawary, M. A. (2017). A semi-analytical iterative method for solving nonlinear thin film flow problems. *Chaos, Solitons & Fractals*, *99*, 52–56. doi:10.1016/j.chaos.2017.03.045

Ayub, M., Rasheed, A., & Hayat, T. (2003). Exact flow of a third grade fluid past a porous plate using homotopy analysis method. *International Journal of Engineering Science*, *41*, 2091–2103. doi:10.1016/S0020-7225(03)00207-6

Bougoffa, L., Duan, J. S., & Rach, R. C. (2015). Exact and approximate analytic solutions of the thin film flow of fourth-grade fluids by the modified Adomian decomposition method. *International Journal of Numerical Methods for Heat & Fluid Flow*, *26*, 2432–2440. doi:10.1108/HFF-07-2015-0278

Daftardar-Gejji, V., & Bhalekar, S. (2009). Solving nonlinear functional equation using Banach contraction principle. *Far East Journal of Applied Mathematics*, *34*(3), 303–314.

Ellahi, R. Tariq, M. H., Hassan, M., & Vafai, K. (2017). On boundary layer nano-ferroliquid flow under the influence of low oscillating stretchable rotating disk. *Journal of Molecular Liquids*, *229*, 339–345. doi:10.1016/j.molliq.2016.12.073

Ellahi, R., Zeeshan, A., & Hassan, M. (2016). Particle shape effects on Marangoni convection boundary layer flow of a nanofluid. *International Journal of Numerical Methods for Heat & Fluid Flow*, *26* (7), 2160–2174. doi:10.1108/HFF-11-2014-0348

Ellahi, R., Zeeshan, A., Shehzad, N., & Alamri, S. Z. (2018). Structural impact of kerosene-Al_2O_3 nanoliquid on MHD Poiseuille flow with variable thermal conductivity: Application of cooling process. *Journal of Molecular Liquids*, *264*, 607–615. doi:10.1016/j.molliq.2018.05.103

He, J. H. (1999). Variational iteration method – a kind of non-linear analytical technique: some examples. *International Journal of Non-linear Mechanics*, *34*(4), 699–708. doi:10.1016/S0020-7462(98)00048-1

Hu, B. & Kieweg, S. L. (2012), The effect of surface tension on the gravity-driven thin film flow of Newtonian and power-law fluids. *Comput Fluids*, *64*, 83–90. doi:10.1016/j.compfluid.2012.05.009

Joshi, M. C. & Bose, R. K. (1985). *Some Topics in Nonlinear Functional Analysis*, Wiley Eastern Limited, New Delhi.

Khan A. A., Masood F., Ellahi R. & Bhatti M. M. (2018). Mass transport on chemicalized fourth-grade fluid propagating peristaltically through a curved channel with magnetic effects. *Journal of Molecular Liquids*, *258*, 186–195. doi:10.1016/j.molliq.2018.02.115

Moosavi, M., Momeni, M., Tavangar, T., Mohammadyari, R. & Rahimi-Esbo, M. (2016), Variational iteration method for flow of non-Newtonian fluid on a moving belt and in a collector. *Alexandria Engineering Journal*, *55*, 1775–1783. doi:10.1016/j.aej.2016.03.033

Odibat, Z. M. (2010). A study on the convergence of variational iteration method. *Mathematical and Computer Modelling*, *51*(9-10), 1181–1192. doi:10.1016/j.mcm.2009.12.034

Sajid, M., & Hayat, T. (2008). The application of homotopy analysis method to thin film flows of a third order fluid. *Chaos, Solitons & Fractals*, *38*, 506–515. doi:10.1016/j.chaos.2006.11.034

Sajid, M., Hayat, T., & Asghar, S. (2007). Comparison between HAM and HPM solutions of thin film flows of non-Newtonian fluids on a moving belt. *Nonlinear Dynamics*, *50*, 27–35. doi:10.1007/s11071-006-9140-y

Siddiqui, A. M., Farooq, A. A., Haroon, T., & Babcock, B. S. (2012). A comparison of variational iteration and Adomian decomposition methods in solving nonlinear thin film flow problems. *Applied Mathematical Sciences*, *6*, 4911–4919. Retrieved from http://www.m-hikari.com/ams/ams-2012/ams-97-100-2012/babcockAMS97-100-201-2.pdf

Siddiqui, A. M., Mahmood, R., & Ghori, Q. K. (2006). Thin film flow of a third grade fluid on a moving belt by He's homotopy perturbation method. *International Journal of Nonlinear Sciences and Numerical Simulation*, *7*(1), 7–14. doi:10.1515/IJNSNS.2006.7.1.7

Sisko, A. W. (1958). The flow of lubrication greases. *Industrial & Engineering Chemistry*, *50*, 1789–1792. doi:10.1021/ie50588a042

Zeeshan, A., Shehzad, N., & Ellahi, R. (2018). Analysis of activation energy in Couette-Poiseuille flow of nanofluid in the presence of chemical reaction and convective boundary conditions. *Results in Physics*, 8, 502–512. doi:10.1016/j.rinp.2017.12.024

Zhaosheng, Y., & Jianzhong, L. (1998). Numerical research on the coherent structure in the viscoelastic second order mixing layers. *Applied Mathematics and Mechanics*, *19*, 717–723. doi:10.1007/BF02457746